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# Equality vs. Equivalence

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**Abstract.** Many differences exist between the logical relations "equality" and "equivalence". In this monograph I point out differences that concern definition, linguistics, computational gates and tables, denotation, application, negation of terms, negation of the relation, relations to other relations, the laws of symmetry, transitivity and reflexivity, the laws of commutation and permutation, the law of tautology, the law of distribution, the law of association, propositional meaning, and "genesis". I also point out a form of "symmetry breaking": the negation of some equality may yield some non-equality, while the negation of that non-equality might not yield the same equality.

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## Introduction

Often I am uncertain about whether to use the symbol for equality $=$ or the symbol for equivalence $\equiv$.

Although many important differences between equality and equivalence have already been defined in the early works on logic (and although since then these relations have been applied and investigated in new ways, see e.g. [14]), a new and more thorough foundational investigation – one that returns to square one, so to speak – might still be welcome.

From a philosophical point of view it is worthwhile to study relations like $=$ and (the connectives that define) $\equiv$ because these seem required to do mathematics, and mathematics is required to do physics, and physics describes reality. I believe that the most basic relations of mathematics are phenomena (not just tools) and that a complete understanding of these relations is required for a complete understanding of reality (down to its smallest possible scale).

This treatise builds on the research of Boole, De Morgan, Frege, Hilbert & Ackermann, Leibniz, Peano, Russell, and Whitehead & Russell [2, 5, 7, 8, 10, 13, 16, 18]. Some familiarity with perhaps these works or any good introduction to logic (or set-theory) is desired.

As said, my aim is to expose the many ways in which $=$ and $\equiv$ are different (aside from being different marks).

First, the general form of any mathematical statement is introduced; it is required to make my exposition general and concise.

Next, a crude account of a theory of denotation is given; some of the differences between equality and equivalence that are pointed out do not hold without it.

Then the main relations of logic are introduced; many differences between equality and equivalence cannot be pointed out without these relations. In fact (topic 1), equality and equivalence are made different because the latter is defined by means of other relations.

Next the paper addresses the following topics with respect to $=$ and $\equiv$

2. linguistics
3. computational gates and tables
4. denotation
5. application
6. negation of terms
7. negation of the relation (a form of "symmetry breaking" is pointed out)
8. relations to other relations
9. symmetry, transitivity and reflexivity
10. commutation and permutation
11. tautology

12. distribution
13. association
14. propositional meaning
15. genesis

The paper concludes with a short overview of the fifteen interdependent differences.

There are five appendices: A concerns the theory of denotation, B concerns topics 1 and 7, C concerns topics 7 and 14, and D concerns appendix C and topic 5; appendix E addresses some criticism I encountered, and helps explain the methodology and relevance of this paper (in the light of contemporary logic).

## 1.1. The Mathematical Statement

Any mathematical statement can be given in the form of $\mathcal{A}\ \mathbf{R}\ \mathcal{B}$ [16]. Here $\mathcal{A}$ and $\mathcal{B}$ are terms, and $\mathbf{R}$ is the relation from the one term to the other (with $\mathcal{A}\ \mathbf{R}\ \mathcal{B}$ equal to $\mathcal{B}\ \mathbf{R}\ \mathcal{A}$ when $\mathbf{R}$ is symmetric and/or commutative and permutative).

For example, the power series $a^0 + a^1x + a^2x^2 + etc$ can be given in the form of $a^0 + z$, with $z = a^1x + a^2x^2 + etc$ .

$\mathcal{A}\ \mathbf{R}\ \mathcal{B}$ is the shortest explicit form of any mathematical statement (the shortest implicit form being a single term $\mathcal{S}$, with $\mathcal{S} = (\mathcal{A}\ \mathbf{R}\ \mathcal{B})$).

When (unlike e.g. $\mathcal{S}$ ) none of the terms in $\mathcal{A}\ \mathbf{R}\ \mathcal{B}$ equals some other (proper) statement, call the terms basic; when the relation is not defined by multiple other relations, call the relation basic; when the terms are basic and the relation is basic, call the statement basic. Any non-basic statement can be written as a combination of only basic terms and only basic relations.

## 1.2. On Denoting

Assume that terms denote "things-in-themselves" [12], i.e. they are symbols that stand for things that are "out there" (like our perceptions stand for things that exist, so to speak, independent from and outside of consciousness).

Let particular statements (i.e. particular non-basic terms) "correspond to" (do not say "denote") "situations" which consist of multiple things-in-themselves that are somehow "connected". The connection(s) somehow involve the relation(s) $\mathbf{R}$ of the statement.

Assume that any particular relation $\mathbf{R}$ of the statement instructs us to select some things-in-itself, or some situation of things-in-themselves, from the situation that corresponds to the statement. In this sense any particular relation $\mathbf{R}$ denotes some "act" of selection (Boole speaks of "election" [2]).

Consequently, any statement with some particular relation denotes some selection (namely from the situation to which the statement corresponds).

For example, the relation $\cup$ in $\mathcal{A} \cup \mathcal{B}$ instructs us to select that which is denoted by $\mathcal{A}$ and/or that which is denoted by $\mathcal{B}$. For example, the relation $\cap$ in $\mathcal{A} \cap \mathcal{B}$ instructs us to select that which is denoted by both $\mathcal{A}$ and $\mathcal{B}$. For example, the relation $=$ in $\mathcal{A} = \mathcal{B}$ instructs us to select that which is denoted by some single term, say $\mathcal{A}$. For example, the relation $\neg$ in $\neg\mathcal{A}$ instructs us to select that which is not denoted by the single term $\mathcal{A}$.

The general statement $\mathcal{A}\ \mathbf{R}\ \mathcal{B}$ (or its negation) denotes the situation to which these particular statements (in which the "variable" $\mathbf{R}$ takes on some specific "value" $\cup$, $\cap$, or $=$) correspond.

Call any particular relation that corresponds to some connection from one thing-in-itself to some other (and that denotes some act of selection with respect to these things-in-themselves) "proper" or "non-trivial".

When some particular relation does not correspond to some connection from one thing-in-itself to some other (and does not denote some act of selection with respect to different things-in-themselves), the relation might be called "trivial" and be treated as a "pseudo-relation". Let any statement in which the particular $\mathbf{R}$ is a pseudo-relation, be equal to (or say "reduce to") one of its terms (or the negation of one of its terms).

Finally, there are properties such as cardinal numbers, ordinal numbers, and truth-values, as well as, for example, lengths, areas, angles, and volumes. Somehow, the things-in-themselves that are selected can be assumed to "posses" such properties, and these properties are not necessarily identical to the things-in-themselves. Note that properties can also be treated as terms in their own right (denoting things-in-themselves in their own right).

The theory of denotation of this section is rather crude; I intend to improve upon it somewhere else. For now, this section and appendix A must do.

## 1.3. The Relation R

The nature of the relation $\mathbf{R}$ depends on the nature of the terms of the statement. For example, if the terms are propositions or sets, then $\mathbf{R}$ may be one of the basic relations

- disjunction; write $\cup$
- implication/inclusion; write $\subset$ (the converse is $\supset$)
- conjunction; write $\cap$

These relations are called connectives. With the exception of implication or inclusion (which Hume argued are identical [11]), connectives are symmetric and commutative and permutative.

Many authors pick one connective and define the others by means of the chosen one and negation (write $\neg$) [8, 10, 18]. Indeed, connectives are related by the well-known identities (named after De Morgan) [7]

$$\begin{array}{lllllll}
(\mathcal{A} \cup \mathcal{B}) & = & (\neg\mathcal{A} \subset \mathcal{B}) & = & (\mathcal{A} \supset \neg\mathcal{B}) & = & \neg(\neg\mathcal{A} \cap \neg\mathcal{B}) \\
(\mathcal{A} \cup \neg\mathcal{B}) & = & (\neg\mathcal{A} \subset \neg\mathcal{B}) & = & (\mathcal{A} \supset \mathcal{B}) & = & \neg(\neg\mathcal{A} \cap \mathcal{B}) \\
(\neg\mathcal{A} \cup \mathcal{B}) & = & (\mathcal{A} \subset \mathcal{B}) & = & (\neg\mathcal{A} \supset \neg\mathcal{B}) & = & \neg(\mathcal{A} \cap \neg\mathcal{B}) \\
(\neg\mathcal{A} \cup \neg\mathcal{B}) & = & (\mathcal{A} \subset \neg\mathcal{B}) & = & (\neg\mathcal{A} \supset \mathcal{B}) & = & \neg(\mathcal{A} \cap \mathcal{B})
\end{array}$$

and

$$\begin{array}{lllllll}
(\mathcal{A} \cap \mathcal{B}) & = & \neg(\neg\mathcal{A} \supset \mathcal{B}) & = & \neg(\mathcal{A} \subset \neg\mathcal{B}) & = & \neg(\neg\mathcal{A} \cup \neg\mathcal{B}) \\
(\mathcal{A} \cap \neg\mathcal{B}) & = & \neg(\neg\mathcal{A} \supset \neg\mathcal{B}) & = & \neg(\mathcal{A} \subset \mathcal{B}) & = & \neg(\neg\mathcal{A} \cup \mathcal{B}) \\
(\neg\mathcal{A} \cap \mathcal{B}) & = & \neg(\mathcal{A} \supset \mathcal{B}) & = & \neg(\neg\mathcal{A} \subset \neg\mathcal{B}) & = & \neg(\mathcal{A} \cup \neg\mathcal{B}) \\
(\neg\mathcal{A} \cap \neg\mathcal{B}) & = & \neg(\mathcal{A} \supset \neg\mathcal{B}) & = & \neg(\neg\mathcal{A} \subset \mathcal{B}) & = & \neg(\mathcal{A} \cup \mathcal{B})
\end{array}$$

I shall use these identities over and over. Remarkably, the identical statements should denote exactly the same selection from the situations of things-in-themselves to which they correspond.

**R** may also be one of the non-basic relations

- equivalence; write $\equiv$
- non-equivalence; write $\not\equiv$

These relations are called non-basic because $\mathcal{A} \equiv \mathcal{B}$ is defined by means of the connectives (and $\mathcal{A} \not\equiv \mathcal{B}$ is simply the negation of $\mathcal{A} \equiv \mathcal{B}$).

One definition of equivalence is

**Definition 1.** $(\mathcal{A} \subset \mathcal{B}) \cap (\mathcal{A} \supset \mathcal{B})$

The first definition of equivalence is used by most authors [8, 10, 15, 18].

Another definition of equivalence is

**Definition 2.** $(\mathcal{A} \cup \mathcal{B}) \subset (\mathcal{A} \cap \mathcal{B})$

The second definition of equivalence is ignored by most authors (except by e.g. Boole, more or less [3]).

The identities given earlier ensure that there are at least sixty-four ways to state either definition (see appendix B). Consider $\mathcal{A} \equiv \mathcal{B}$ a shortcut to writing any of these statements.

The law of distribution can be used to show that the two definitions are equal.

Instead of the non-basic $\not\equiv$, one might use the basic relation

- non-equality; write $\neq$

Non-equality is somewhat elusive since it is the negation of

- equality; write $=$

In the statement $\mathcal{A} = \mathcal{B}$, both terms denote the same thing-in-itself; so seen there is only a single term, not some statement in which one term is related to some other (see also [8, 17]). From this point of view, $=$ is a pseudo-relation and $\mathcal{A} = \mathcal{B}$ is a "pseudo-statement".

Another pseudo-relation that concerns only one term is

- negation; write $\neg$

A term is negated by placing the symbol $\neg$ in front of it ($\neg\mathcal{A}$ is used instead of $\mathcal{A}\neg$, and $\mathcal{A} \neg \mathcal{B}$ is not used). Similarly, a statement is negated by placing the symbol $\neg$ in front of it. Let a relation be negated by negating the statement in which it occurs, so that $(\mathcal{A} \neg\mathbf{R}\, \mathcal{B}) = \neg(\mathcal{A}\ \mathbf{R}\ \mathcal{B})$.

(Perhaps consider $\neg$ equal to the relation $\neq$ applied to a single term $\mathcal{A}$, and the usual $\neq$ equal to the relation $\neg$ as in $\mathcal{A} \neg \mathcal{B}$ ).

The negated term is just another term, and the negated statement is just another statement. Given $\mathcal{A}\ \mathbf{R}\ \mathcal{B}$ , application of negation yields one out of seven other statements (namely $\mathcal{A}\ \mathbf{R}\ \neg\mathcal{B}$, $\neg\mathcal{A}\ \mathbf{R}\ \mathcal{B}$, $\neg\mathcal{A}\ \mathbf{R}\ \neg\mathcal{B}$, or any of their negations, or the negation of $\mathcal{A}\ \mathbf{R}\ \mathcal{B}$ ).

When $\neg$ is placed in front of some negated term or statement, the original term or statement is again obtained, i.e. $\neg\neg\mathcal{A}$ equals $\mathcal{A}$. This is the law of double negation.

Another law concerns the conjunction of a term and the negation of that term; it states that $\neg\mathcal{A} \cap \mathcal{A}$ denotes nothing and is equal to the empty-term $\emptyset$ (which in any disjunction is omissible, i.e. $\mathcal{A} \cup \emptyset = \mathcal{A}$ ).

The negation of $\emptyset$ equals the universal term $\mathbf{U}$, which then equals $\mathcal{A} \cup \neg\mathcal{A}$ .

## 2. Linguistics, $=$ vs. $\equiv$

Disjunction ($\cup$) equals the linguistic connective "... and/or ...", as in the noun-phrase "this and/or that". In addition, any statement in which $\mathbf{R}$ is $\cup$ equals a phrase with a verb like "this and that are disjoined".

Implication/inclusion ($\subset$) equals the preposition(s) "... so/in ...", as in the noun-phrase "this so/in that". In addition, any statement in which $\mathbf{R}$ is $\subset$ equals a phrase with a verb like "this implies/is included in that" or "that is implied by/includes this".

Conjunction ($\cap$) equals the linguistic connective "both ... and ...", as in the noun-phrase "both this and that". In addition, any statement in which $\mathbf{R}$ is $\cap$ equals a phrase with a verb like "this and that are conjoined".

Negation ($\neg$) equals the linguistic adjective/adverb/preposition "not", as in the noun-phrase "not this", or as in the phrase with a verb "this is not".

Perhaps consider noun-phrases timeless and space-like, and, like Aristotle [1], assume that any verb defines some "mode of being" or "action" in time (perhaps requiring some kind of agency).

### 2.1. Equality

Equality does not equal some linguistic connective or preposition, and no statement in which **R** is $=$ equals a noun-phrase (unless the statement is treated as a pseudo-statement equal to a single term, which usually equals a noun or noun-phrase).

Any statement in which **R** is $=$ equals a phrase with a verb, namely "to equal" or "to be" or "to be identical to", like "this equals that" or "this is that" or "this is identical to that".

### 2.2. Equivalence

Equivalence does not directly equal some linguistic connective or preposition.

That said, by the first and second definition of equivalence, any statement in which **R** is $\equiv$ equals some statement in which only logical connectives occur, and therefore equals a phrase in which only linguistic connectives and prepositions occur.

In addition, any statement in which **R** is $\equiv$ equals a phrase with a verb, namely "to be equivalent to", like "this is equivalent to that".

## 3. Computational Gates and Tables, $=$ vs. $\equiv$

Any statement in which **R** is one of the logical connectives (disjunction, implication/inclusion, or conjunction) can be "implemented" by some computational gate that counts two inputs and one output; in fact, any computational gate in any classical computer may be defined in terms of such gates.

In addition, any computational gate is the implementation of some computational table. For example, let the conjunction $\mathcal{A} \cap \mathcal{B}$ equal the computational table

| | $\mathcal{B}$ | $\neg\mathcal{B}$ |
|---|---|---|
| $\mathcal{A}$ | *yes* | *no* |
| $\neg\mathcal{A}$ | *no* | *no* |

$$= (\mathcal{A} \cap \mathcal{B}) \cup \cancel{(\mathcal{A} \cap \neg\mathcal{B})} \cup \cancel{(\neg\mathcal{A} \cap \mathcal{B})} \cup \cancel{(\neg\mathcal{A} \cap \neg\mathcal{B})}$$

### 3.1. Equality

Since the terms in the statement $\mathcal{A} = \mathcal{B}$ are equal, $\mathcal{A} = \mathcal{B}$ cannot be implemented by some computational gate that counts two different inputs and one output. Instead, the statement $\mathcal{A} = \mathcal{B}$, or simply the term $\mathcal{A}$, or simply the term $\mathcal{B}$, is implemented by some (pseudo-)gate that counts one input and one output equal to that input, i.e. a buffer-gate or simply a wire.

The computational table equal to $\mathcal{A} = \mathcal{B}$ is

| | $\mathcal{B}=\mathcal{A}$ | $\neg\mathcal{B}=\neg\mathcal{A}$ |
|---|---|---|
| $\mathcal{A}=\mathcal{B}$ | *yes* | *no* |
| $\neg\mathcal{A}=\neg\mathcal{B}$ | *no* | *no* |

equal to

| | |
|---|---|
| $\mathcal{A}$ | *yes* |
| $\neg\mathcal{A}$ | *no* |

equal to

| | |
|---|---|
| $\mathcal{B}$ | *yes* |
| $\neg\mathcal{B}$ | *no* |

Similarly, $\neg\mathcal{A} = \neg\mathcal{B}$ or simply $\neg\mathcal{A}$ or simply $\neg\mathcal{B}$, is equal to

| | $\mathcal{B}=\mathcal{A}$ | $\neg\mathcal{B}=\neg\mathcal{A}$ |
|---|---|---|
| $\mathcal{A}=\mathcal{B}$ | *no* | *no* |
| $\neg\mathcal{A}=\neg\mathcal{B}$ | *no* | *yes* |

equal to

| | |
|---|---|
| $\mathcal{A}$ | *no* |
| $\neg\mathcal{A}$ | *yes* |

equal to

| | |
|---|---|
| $\mathcal{B}$ | *no* |
| $\neg\mathcal{B}$ | *yes* |

(Recall that $\neg\mathcal{A} \cap \mathcal{A}$ denotes nothing and can be omitted.)

~~Note that~~ computational gates that represent the connectives (i.e. switches in parallel or series) can (once the switches have been put in some state) transmit a signal from a single source of power, while buffer-gates (=) and inverters ($\neg$) use an additional power-source to transmit ~~signals~~. [SCRAP]

### 3.2. Equivalence

By the first definition of equivalence, the statement $\mathcal{A} \equiv \mathcal{B}$ can be implemented by the combination of computational gates of figure 1.

By the second definition of equivalence, the statement $\mathcal{A} \equiv \mathcal{B}$ can be implemented by the combination of computational gates of figure 2.

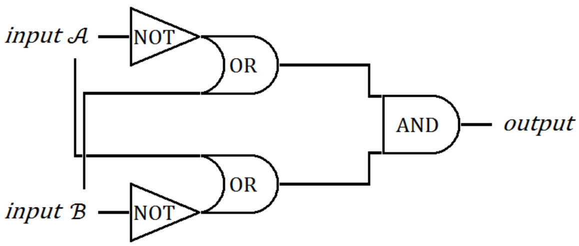


FIGURE 1. the first definition of equivalence in computational gates

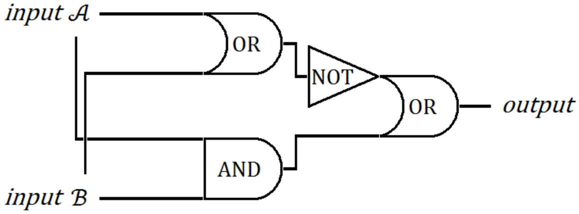


FIGURE 2. the second definition of equivalence in computational gates

Both combinations are implementations of the computational table

| | $\mathcal{B}$ | $\neg\mathcal{B}$ |
|---|---|---|
| $\mathcal{A}$ | *yes* | *no* |
| $\neg\mathcal{A}$ | *no* | *yes* |

$$= (\mathcal{A} \cap \mathcal{B}) \cup \cancel{(\mathcal{A} \cap \neg\mathcal{B})} \cup \cancel{(\neg\mathcal{A} \cap \mathcal{B})} \cup (\neg\mathcal{A} \cap \neg\mathcal{B})$$

This statement is equal to $(\mathcal{A} \cap \mathcal{B}) \cup (\neg\mathcal{A} \cap \neg\mathcal{B})$, equal to the second definition of equivalence $(\mathcal{A} \cap \mathcal{B}) \supset (\mathcal{A} \cup \mathcal{B})$.

Compare this to the negation of $\mathcal{A} \equiv \mathcal{B}$, equal to the computational table

| | $\mathcal{B}$ | $\neg\mathcal{B}$ |
|---|---|---|
| $\mathcal{A}$ | *no* | *yes* |
| $\neg\mathcal{A}$ | *yes* | *no* |

$$= \cancel{(\mathcal{A} \cap \mathcal{B})} \cup (\mathcal{A} \cap \neg\mathcal{B}) \cup (\neg\mathcal{A} \cap \mathcal{B}) \cup \cancel{(\neg\mathcal{A} \cap \neg\mathcal{B})}$$

## 4. Denotation, = vs. ≡

### 4.1. Equality

When the symbols $\mathcal{A}$ and $\mathcal{B}$ denote the same thing-in-itself (say, $\mathbf{x}$), they can be used instead of one another; this fact is expressed by the statement $\mathcal{A} = \mathcal{B}$.

Note that would the same symbols denote themselves as things-in-themselves (say, $\mathbf{a}$ and $\mathbf{b}$), then $\mathcal{A} \neq \mathcal{B}$; after all, $\mathcal{A}$ and $\mathcal{B}$ are different letters.

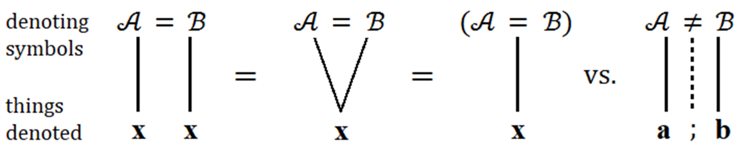


FIGURE 3. the symbols $\mathcal{A}$ and $\mathcal{B}$, equal versus non-equal

The relation equality does not correspond to some connection from the thing-in-itself denoted by $\mathcal{A}$ to the thing-in-itself denoted by $\mathcal{B}$; in fact, the statement $\mathcal{A} = \mathcal{B}$ denotes the same thing-in-itself as $\mathcal{A}$ alone or $\mathcal{B}$ alone. In this sense, the relation equality (unlike any proper/non-trivial relation) purely concerns the symbols that denote, not the things-in-themselves denoted (see also [8, 17]).

Of course, with respect to the symbols that denote, = behaves like a proper relation, and $\mathcal{A} = \mathcal{B}$ (though perhaps not $\mathcal{A} = \mathcal{A}$, see figure 4) like a proper statement. However, with respect to the things-in-themselves, = is a pseudo-relation and $\mathcal{A} = \mathcal{B}$ is a pseudo-statement equal to a single term.

It makes sense to treat = as a pseudo-relation, and $\mathcal{A} = \mathcal{B}$ as a pseudo-statement, when equality is compared to some particular $\mathbf{R}$ that corresponds to some connection from one thing-in-itself to some other, i.e. when equality is placed on the same footing as such (proper/non-trivial) $\mathbf{R}$.

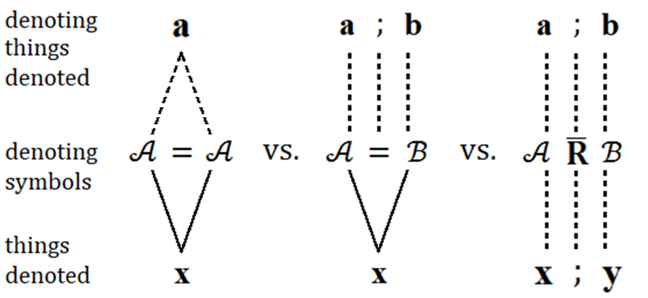


FIGURE 4. the symbol $\mathcal{A}$ equal to $\mathcal{A}$, versus equal to $\mathcal{B}$, versus (properly) related to $\mathcal{B}$

It makes sense to not treat = as a pseudo-relation when $\mathcal{A} = \mathcal{B}$ is compared to some other statement in which the relation **R** is also equality. For example, let us not treat equality as a pseudo-relation in "both $\mathcal{A} = \mathcal{B}$ and $\mathcal{B} = \mathcal{C}$, so/in $\mathcal{A} = \mathcal{C}$".

### 4.2. Equivalence

By either definition of equivalence, any equivalence equals some statement with only connectives (of which at least some are non-trivial); any non-trivial connective corresponds to some connection from one thing-in-itself to some other; therefore $\equiv$ indirectly corresponds to some connection(s) from one thing-in-itself to some other.

TABLE 1. similarities and differences between the relations of logic

| **R** | $\cup$ | $\subset$ | $\cap$ | $\equiv$ | $\not\equiv$ | $\neq$ | $=$ | $\neg$ |
|---|---|---|---|---|---|---|---|---|
| *corresponds (proper- -ly) to connection* | *yes*$^*$ | *yes*$^*$ | *yes*$^*$ | *yes*$^\dagger$ | *yes*$^\dagger$ | *yes?*$^\ddagger$ | *no* | *no?* |

$^*$mind tautology, see 11; $^\dagger$namely indirectly; $^\ddagger$see 7, A and C.

## 5. Application, = vs. $\equiv$

### 5.1. Equality

When different symbols can be used instead of others, equality comes into play.

There are two possibilities: either the replacements are allowed only at certain points in space and/or time, or they are allowed everywhere and always.

When the first possibility is the case, one of the symbols is called a variable, and the other a value (which might be another variable). For example, at one point in space and/or time the variable $f(x)$ may equal the value $f(0)$, while at another point in space and/or time it may equal the value $f(1)$.

When the second possibility is the case, one of the symbols is called the definiendum (that which is defined) and the other the definition. For example, the definiendum $\mathcal{A} \equiv \mathcal{B}$ can everywhere and always be replaced by the first definition $(\mathcal{A} \subset \mathcal{B}) \cap (\mathcal{A} \supset \mathcal{B})$, and it can everywhere and always be replaced by the second definition $(\mathcal{A} \cup \mathcal{B}) \subset (\mathcal{A} \cap \mathcal{B})$; since both definitions can everywhere and always replace $\mathcal{A} \equiv \mathcal{B}$, they also define one another.

Apparently, a term or statement can have multiple definitions. For example the number 6 can be defined as $1+2+3$ or as $1 \bullet 2 \bullet 3$ or as $2 \bullet 3$ or as $12 \div 2$ or as $(10 \bullet 6) \div 10$, and so on, ad infinitum (though we might like to avoid "reflexive" definitions, like the latter two).

To emphasize that $=$ relates some variable to some value, we could write $=_v$; to emphasize that $=$ relates some definiendum to some definition, we could write $=_{df}$.

For example, let $b2 = p1$ mean "buy two, pay one"; then as $b2 =_v p1$, the discount is given somewhere for some time, and as $b2 =_{df} p1$, the discount is given everywhere and always.

### 5.2. Equivalence

Equivalence is used to state that denoting terms share certain properties (see e.g. [4] and [9]).

To emphasize this use, we could write $\mathcal{A} \equiv_{\mathfrak{p}} \mathcal{B}$ instead of $\mathcal{A} \equiv \mathcal{B}$, and we could write the first definition as $(\mathcal{A} \subset_{\mathfrak{p}} \mathcal{B}) \cap (\mathcal{A} \supset_{\mathfrak{p}} \mathcal{B})$, and the second definition as $(\mathcal{A} \cup_{\mathfrak{p}} \mathcal{B}) \subset (\mathcal{A} \cap_{\mathfrak{p}} \mathcal{B})$. Obviously, the subscript $\mathfrak{p}$ stands for "property" and indicates that the relation holds with respect to some property.

For example, let $\mathcal{A} \equiv_{\mathfrak{c}} \mathcal{B}$ mean that $\mathcal{A}$ and $\mathcal{B}$ share the same cardinal number. For example, let $\mathcal{A} \equiv_{\mathfrak{o}} \mathcal{B}$ mean that $\mathcal{A}$ and $\mathcal{B}$ share the same ordinal number. For example, let $\mathcal{A} \equiv_{T\backslash F} \mathcal{B}$ mean that $\mathcal{A}$ and $\mathcal{B}$ share the same truth-value. For example, let $\mathcal{A} \equiv_{congruent} \mathcal{B}$ mean that $\mathcal{A}$ and $\mathcal{B}$ are geometrically congruent. Often $\equiv_{\mathfrak{p}}$ is replaced by some other symbol, for example $\equiv_{congruent}$ by the symbol $\cong$.

Equal terms should have all possible properties in common, and therefore be equivalent in every possible way. However, terms that are equivalent in one or more ways do not have to be equal; they may denote different things-in-themselves.

It makes sense to consider $\mathcal{A} =_{\mathfrak{p}} \mathcal{B}$ equal to $\mathcal{A} \equiv_{\mathfrak{p}} \mathcal{B}$, and to consider $\mathcal{A}_{\mathfrak{p}} \equiv \mathcal{B}_{\mathfrak{p}}$ equal to $\mathfrak{p} \equiv \mathfrak{p}$, and to consider $\mathcal{A}_{\mathfrak{p}} = \mathcal{B}_{\mathfrak{p}}$ equal to $\mathfrak{p} = \mathfrak{p}$.

# 6. Negation of Terms, = vs. ≡

## 6.1. Equality

Take $\mathcal{A} = \mathcal{B}$ and negate one or both terms to obtain $\mathcal{A} = \neg\mathcal{B}$ or $\neg\mathcal{A} = \mathcal{B}$ or $\neg\mathcal{A} = \neg\mathcal{B}$. Since $\mathcal{A}$ and $\neg\mathcal{A}$ are non-equal, and $\mathcal{B}$ and $\neg\mathcal{B}$ are non-equal, the four statements are non-equal.

## 6.2. Equivalence

Take $\mathcal{A} \equiv \mathcal{B}$ and negate one or both terms to obtain $\mathcal{A} \equiv \neg\mathcal{B}$ or $\neg\mathcal{A} \equiv \mathcal{B}$ or $\neg\mathcal{A} \equiv \neg\mathcal{B}$. Interestingly, by either of the two definitions of equivalence, $\mathcal{A} \equiv \mathcal{B}$ equals $\neg\mathcal{A} \equiv \neg\mathcal{B}$, while $\mathcal{A} \equiv \neg\mathcal{B}$ equals $\neg\mathcal{A} \equiv \mathcal{B}$.

Indeed, by the first definition of equivalence, $\mathcal{A} \equiv \mathcal{B}$ equals
$(\mathcal{A} \subset \mathcal{B}) \cap (\mathcal{A} \supset \mathcal{B})$, equal to
$(\neg\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup \neg\mathcal{B})$, equal to
$(\neg\mathcal{A} \supset \neg\mathcal{B}) \cap (\neg\mathcal{A} \subset \neg\mathcal{B})$, equal to
the first definition of the equivalence $\neg\mathcal{A} \equiv \neg\mathcal{B}$.

Indeed, by the second definition of equivalence, $\mathcal{A} \equiv \mathcal{B}$ equals
$(\mathcal{A} \cup \mathcal{B}) \subset (\mathcal{A} \cap \mathcal{B})$, equal to
$\neg(\mathcal{A} \cup \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{B})$, equal to
$(\neg\mathcal{A} \cap \neg\mathcal{B}) \cup (\mathcal{A} \cap \mathcal{B})$, equal to
$(\neg\mathcal{A} \cap \neg\mathcal{B}) \supset \neg(\mathcal{A} \cap \mathcal{B})$, equal to
$(\neg\mathcal{A} \cap \neg\mathcal{B}) \supset (\neg\mathcal{A} \cup \neg\mathcal{B})$, equal to
the second definition of the equivalence $\neg\mathcal{A} \equiv \neg\mathcal{B}$.

Similarly, by the first definition of equivalence, $\mathcal{A} \equiv \neg\mathcal{B}$ equals
$(\mathcal{A} \subset \neg\mathcal{B}) \cap (\mathcal{A} \supset \neg\mathcal{B})$, equal to
$(\neg\mathcal{A} \cup \neg\mathcal{B}) \cap (\mathcal{A} \cup \mathcal{B})$, equal to
$(\neg\mathcal{A} \supset \mathcal{B}) \cap (\neg\mathcal{A} \subset \mathcal{B})$, equal to
the first definition of the equivalence $\neg\mathcal{A} \equiv \mathcal{B}$.

Similarly, by the second definition of equivalence, $\mathcal{A} \equiv \neg\mathcal{B}$ equals
$(\mathcal{A} \cup \neg\mathcal{B}) \subset (\mathcal{A} \cap \neg\mathcal{B})$, equal to
$\neg(\mathcal{A} \cup \neg\mathcal{B}) \cup (\mathcal{A} \cap \neg\mathcal{B})$, equal to
$(\neg\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \neg\mathcal{B})$, equal to
$(\neg\mathcal{A} \cap \mathcal{B}) \supset \neg(\mathcal{A} \cap \neg\mathcal{B})$, equal to
$(\neg\mathcal{A} \cap \mathcal{B}) \supset (\neg\mathcal{A} \cup \mathcal{B})$, equal to
the second definition of the equivalence $\neg\mathcal{A} \equiv \mathcal{B}$.

However $(\mathcal{A} \equiv \mathcal{B}) = (\neg\mathcal{A} \equiv \neg\mathcal{B})$ and $(\mathcal{A} \equiv \neg\mathcal{B}) = (\neg\mathcal{A} \equiv \mathcal{B})$ are non-equal, for the one is the negation of the other (see below).

## 7. Negation of the Relation, = vs. ≡

### 7.1. Equality

Let non-equality ($\neq$) equal the negation of equality ($\neg =$), and assume that to say that $\neg =$ equals $\neq$, is to say that $\neg(\mathcal{A} = \mathcal{B})$ equals $\mathcal{A} \neq \mathcal{B}$ (on the other hand, the negation of e.g. $\mathcal{A} = \mathcal{B} = \mathcal{C}$ might equal either $\mathcal{A} = \mathcal{B} \neq \mathcal{C}$ or $\mathcal{A} \neq \mathcal{B} = \mathcal{C}$ or $\mathcal{C} \neq \mathcal{A} = \mathcal{B}$ or $\mathcal{A} \neq \mathcal{B} \neq \mathcal{C}$, at least if the equality is considered non-trivial).

Now $\neg(\mathcal{A} = \mathcal{B})$ should only equal $\mathcal{A} \neq \mathcal{B}$ when $=$ behaves like a proper relation, and $\mathcal{A} = \mathcal{B}$ like a proper statement (i.e. when $=$ is not a pseudo-relation, and $\mathcal{A} = \mathcal{B}$ is not a pseudo-statement). Then $\mathcal{A} \neq \mathcal{B}$ should mean that the symbols $\mathcal{A}$ and $\mathcal{B}$ denote different things-in-themselves.

When equality is considered a pseudo-relation, so that $\mathcal{A} = \mathcal{B}$ is a pseudo-statement, then the negation $\neg(\mathcal{A} = \mathcal{B})$ should equal the single term $\neg\mathcal{A}$, as well as $\neg\mathcal{B}$, with $\neg\mathcal{A} = \neg\mathcal{B}$.

Let $\neg(\mathcal{A} = \mathcal{B})$ equal $\mathcal{A} \neq \mathcal{B}$; a question is, does $\neq$ purely concern symbols, or does it (also) correspond to some connection between things-in-themselves? Imagine the following

In the beginning there is only one thing-in-itself and there are two different symbols that denote this thing-in-itself, namely $\mathcal{A}$ and $\mathcal{B}$. Then at first $\mathcal{A} = \mathcal{B}$ is consistent, and the negation $\neg(\mathcal{A} = \mathcal{B})$ is either a contradiction or equal to $\neg\mathcal{A}$, as well as to $\neg\mathcal{B}$. Next, let the thing-in-itself split up into two different things-in-themselves, so that $\mathcal{A} = \mathcal{B}$ becomes a contradiction, and ought to be replaced by $\mathcal{A} \neq \mathcal{B}$, assumed equal to $\neg(\mathcal{A} = \mathcal{B})$. Now imagine two possibilities, namely either the two things-in-themselves merge into one again, or they remain different. In the first case, $\mathcal{A} \neq \mathcal{B}$ becomes a contradiction, and ought to be replaced by $\mathcal{A} = \mathcal{B}$, assumed equal to $\neg(\mathcal{A} \neq \mathcal{B})$. In the second case, $\mathcal{A} \neq \mathcal{B}$ is consistent, and the negation $\neg(\mathcal{A} \neq \mathcal{B})$ may equal any contradiction, for example the conjunction $((\mathcal{A} = \mathcal{B}) \cap (\neg\mathcal{A} = \neg\mathcal{B})) = \emptyset$.

Interestingly, to say that a contradictive $\mathcal{A} \neq \mathcal{B}$ yields $\mathcal{A} = \mathcal{B}$ might be problematic. The reason is that when $\mathcal{A} \neq \mathcal{B}$ is a contradiction, it will equal $\emptyset$, and the negation of $\emptyset$ is the universal term $\mathbf{U}$ which is of the form $(\mathcal{A} = \mathcal{B}) \cup (\neg\mathcal{A} = \neg\mathcal{B})$, which does not reduce to a single term unless either $\mathcal{A} = \mathcal{B}$ or $\neg\mathcal{A} = \neg\mathcal{B}$ is $\emptyset$. In this sense, once $\mathcal{A} \neq \mathcal{B}$ is given, there might be no way back to $\mathcal{A} = \mathcal{B}$ (there is a "breaking of symmetry").

Similarly, to say that a contradictive $\mathcal{A} = \mathcal{B}$ yields $\mathcal{A} \neq \mathcal{B}$ might be problematic. The reason is that when $\mathcal{A}$ and $\mathcal{B}$ are only partly non-equal, part or all of $\neg\mathcal{A}$ in $(\mathcal{A} \neq \mathcal{B}) = (\mathcal{A} \cup \neg\mathcal{A})$ has to be $\emptyset$.

Finally, note that neither $\mathcal{A} = \neg\mathcal{B}$ nor $\neg\mathcal{A} = \mathcal{B}$ equals $\mathcal{A} \neq \mathcal{B}$, and compare this to $(\mathcal{A} \equiv \neg\mathcal{B}) = (\neg\mathcal{A} \equiv \mathcal{B}) = (\mathcal{A} \not\equiv \mathcal{B})$ (see below).

### 7.2. Equivalence

Let non-equivalence ($\not\equiv$) equal the negation of equivalence ($\neg \equiv$), and assume that to say that $\neg \equiv$ equals $\not\equiv$ is to say that $\neg(\mathcal{A} \equiv \mathcal{B})$ equals $\mathcal{A} \not\equiv \mathcal{B}$ (on the other hand, with topic 10 it is shown that the negation of $\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C}$ involves only one $\not\equiv$, for $\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C}$ and $\mathcal{A} \not\equiv \mathcal{B} \not\equiv \mathcal{C}$ are equal).

Since $\mathcal{A} \equiv \mathcal{B}$ and $\neg\mathcal{A} \equiv \neg\mathcal{B}$ are equal, $\neg(\mathcal{A} \equiv \mathcal{B})$ and $\neg(\neg\mathcal{A} \equiv \neg\mathcal{B})$ are equal; consequently $\mathcal{A} \not\equiv \mathcal{B}$ and $\neg\mathcal{A} \not\equiv \neg\mathcal{B}$ are equal.

Since $\mathcal{A} \equiv \neg\mathcal{B}$ and $\neg\mathcal{A} \equiv \mathcal{B}$ are equal, $\neg(\mathcal{A} \equiv \neg\mathcal{B})$ and $\neg(\neg\mathcal{A} \equiv \mathcal{B})$ are equal; consequently $\mathcal{A} \not\equiv \neg\mathcal{B}$ and $\neg\mathcal{A} \not\equiv \mathcal{B}$ are equal.

Moreover, e.g. $\mathcal{A} \equiv \mathcal{B}$ is the negation of e.g. $\mathcal{A} \equiv \neg\mathcal{B}$, and vice versa; hence $\mathcal{A} \equiv \mathcal{B}$ equals $\mathcal{A} \not\equiv \neg\mathcal{B}$, while $\mathcal{A} \equiv \neg\mathcal{B}$ equals $\mathcal{A} \not\equiv \mathcal{B}$.

In fact, when $\mathcal{A} \equiv \mathcal{B}$ (as well as $\neg\mathcal{A} \equiv \neg\mathcal{B}$) is defined by the first definition of equivalence, its negation can be put equal to $\mathcal{A} \equiv \neg\mathcal{B}$ (as well as to $\neg\mathcal{A} \equiv \mathcal{B}$) as defined by the second definition of equivalence, and vice versa. Likewise, when $\mathcal{A} \equiv \mathcal{B}$ (as well as $\neg\mathcal{A} \equiv \neg\mathcal{B}$) is defined by the second definition of equivalence, its negation can be put equal to $\mathcal{A} \equiv \neg\mathcal{B}$ (as well as to $\neg\mathcal{A} \equiv \mathcal{B}$) as defined by the first definition of equivalence, and vice versa.

Indeed, by the first definition, e.g. $\neg(\mathcal{A} \equiv \mathcal{B})$ equals
$\neg((\mathcal{A} \subset \mathcal{B}) \cap (\mathcal{A} \supset \mathcal{B}))$, equal to
$\neg((\neg\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup \neg\mathcal{B}))$, equal to
$(\mathcal{A} \cap \neg\mathcal{B}) \cup (\neg\mathcal{A} \cap \mathcal{B})$, equal to both
$(\mathcal{A} \cap \neg\mathcal{B}) \supset \neg(\neg\mathcal{A} \cap \mathcal{B})$ and $\neg(\mathcal{A} \cap \neg\mathcal{B}) \subset (\neg\mathcal{A} \cap \mathcal{B})$, i.e. to
$(\mathcal{A} \cap \neg\mathcal{B}) \supset (\mathcal{A} \cup \neg\mathcal{B})$ and $(\neg\mathcal{A} \cup \mathcal{B}) \subset (\neg\mathcal{A} \cap \mathcal{B})$, i.e. to
the second definition of $\mathcal{A} \equiv \neg\mathcal{B}$ and the second definition of $\neg\mathcal{A} \equiv \mathcal{B}$.

Indeed, by the second definition, e.g. $\neg(\mathcal{A} \equiv \mathcal{B})$ equals
$\neg((\mathcal{A} \cup \mathcal{B}) \subset (\mathcal{A} \cap \mathcal{B}))$, equal to
$\neg(\neg(\mathcal{A} \cup \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{B}))$, equal to
$(\mathcal{A} \cup \mathcal{B}) \cap \neg(\mathcal{A} \cap \mathcal{B})$, equal to
$(\mathcal{A} \cup \mathcal{B}) \cap (\neg\mathcal{A} \cup \neg\mathcal{B})$, equal to both
$(\mathcal{A} \supset \neg\mathcal{B}) \cap (\mathcal{A} \subset \neg\mathcal{B})$ and $(\neg\mathcal{A} \subset \mathcal{B}) \cap (\neg\mathcal{A} \supset \mathcal{B})$, i.e. to
the first definition of $\mathcal{A} \equiv \neg\mathcal{B}$ and the first definition of $\neg\mathcal{A} \equiv \mathcal{B}$.

Let us call the first definition of equivalence "of the $\cap$-kind" (because the middle relation is conjunction); let us call the second definition of equivalence "of the $\cup$-kind" (because the middle relation can be disjunction). Then $\mathcal{A} \equiv \mathcal{B}$ as defined by a definition of the $\cap$-kind directly negates $\mathcal{A} \not\equiv \mathcal{B}$ as defined by a definition of the $\cup$-kind, and vice versa. Likewise, $\mathcal{A} \equiv \mathcal{B}$ as defined by a definition of the $\cup$-kind directly negates $\mathcal{A} \not\equiv \mathcal{B}$ as defined by a definition of the $\cap$-kind, and vice versa. See also appendix B.

## 8. Relations to Other Relations, $=$ vs. $\equiv$

Assume that when **R** relates relations (as in R **R** R or as in R **R** ¬R or as in ¬R **R** R or as in ¬R **R** ¬R), it relates some statement $\mathcal{V}$ **R** $\mathcal{W}$ or $\neg(\mathcal{V}$ **R** $\mathcal{W})$ with the one relation to some statement $\mathcal{X}$ **R** $\mathcal{Y}$ or $\neg(\mathcal{X}$ **R** $\mathcal{Y})$ with the other relation; with $\mathcal{X}$ equal to either $\mathcal{V}$ or $\neg\mathcal{V}$, and $\mathcal{Y}$ equal to either $\mathcal{W}$ or $\neg\mathcal{W}$.

For example, let $\cup \neq \cap$ be equal to $(\mathcal{V} \cup \mathcal{W}) \neq (\neg\mathcal{V} \cap \neg\mathcal{W})$. For example, let $\subset = \supset$ be equal to $(\mathcal{V} \subset \mathcal{W}) = (\neg\mathcal{V} \supset \neg\mathcal{W})$.

### 8.1. Equality

Compare $\mathcal{A} = \mathcal{B}$ with some $\mathcal{A}$ **R** $\mathcal{B}$ in which the relation **R** is some non-trivial $\cup$, $\subset$ or $\cap$, or (non-)equivalence.

To do so is to place $\mathcal{A} = \mathcal{B}$ on the same footing as some statement in which **R** corresponds to some connection from one thing-in-itself to some other, so that $\mathcal{A} = \mathcal{B}$ is a pseudo-statement and $=$ should disappear.

If $=$ disappears as soon as it is related to any **R** that corresponds to some connection from one thing-in-itself to some other, it cannot be related to such **R**.

For example, to state that $= \subset \equiv$ (i.e. equality so/in equivalence) is to state that $(\mathcal{V} = \mathcal{W}) \subset (\mathcal{X} \equiv \mathcal{Y})$, which is to state that $\mathcal{V} \subset (\mathcal{X} \equiv \mathcal{Y})$ as well as $\mathcal{W} \subset (\mathcal{X} \equiv \mathcal{Y})$; hence $=$ is not related to $\equiv$ as intended.

Similarly, neither $= \supset \equiv$ (i.e. equivalence so/in equality) nor $= \equiv \equiv$ (i.e. equality is equivalent to equivalence) nor $= = \equiv$ (i.e. equality equals equivalence) seems to relate equality to equivalence as attempted. Worse, $= \neq \equiv$ might not relate two relations.

On the other hand, it seems that $=$ can be related to itself, as in $((\mathcal{A} = \mathcal{B}) \cap (\mathcal{B} = \mathcal{C})) \subset (\mathcal{A} = \mathcal{C})$, see topic 9 (compare this to $(\mathcal{A} \cap \mathcal{A}) \subset \mathcal{A}$, see topic 11); it also seems that $=$ can be related to some trivial connective, see topic 11. In addition, $=$ and $\neq$ seem to negate one another, see topic 7.

Now, how is it that we can compare equality and equivalence, and speak of differences, if $= \neq \equiv$ does not relate relations? One might reason that if $= \neq \equiv$ does not work, then $= = \equiv$; on the other hand, since $=$ and $\equiv$ behave differently, $= = \equiv$ does not work, so that $= \neq \equiv$. Accordingly, equality and equivalency are equal and non-equal to one another, and therefore neither equal nor non-equal to one another. Can this be true and consistent, or is this false and a contradiction? The same logic applies to equality and the other non-trivial/proper relations ($\neq$ included?).

### 8.2. Equivalence

When **R** is equivalence, it is readily related to the connectives, and to itself, and to non-equivalence; see the first and second definition (topic 1), as well as topic 6 and topic 7.

## 9. Symmetry, Transitivity and Reflexivity, $=$ vs. $\equiv$

It makes sense to say that

### 9.1. Equality

Equality is symmetric: $\mathcal{A} = \mathcal{B}$ equals $\mathcal{B} = \mathcal{A}$.

Non-equality is also symmetric: $\mathcal{A} \neq \mathcal{B}$ equals $\mathcal{B} \neq \mathcal{A}$.

Equality is transitive: if $\mathcal{A} = \mathcal{B}$ and $\mathcal{B} = \mathcal{C}$, then $\mathcal{A} = \mathcal{C}$; alternatively state: if $\mathcal{A} = \mathcal{B} = \mathcal{C}$, then $\mathcal{A} = \mathcal{C}$.

Non-equality is not transitive: if $\mathcal{A} \neq \mathcal{B}$ and $\mathcal{B} \neq \mathcal{C}$, then either $\mathcal{A} \neq \mathcal{C}$ or $\mathcal{A} = \mathcal{C}$.

Equality is reflexive: $\mathcal{A} = \mathcal{A}$ is consistent. Also, if $\mathcal{A} = \mathcal{B}$ and $\mathcal{B} = \mathcal{A}$, then $\mathcal{A} = \mathcal{A}$.

Non-equality is not reflexive: $\mathcal{A} \neq \mathcal{A}$ is a contradiction.

### 9.2. Equivalence

Equivalence is symmetric: $\mathcal{A} \equiv \mathcal{B}$ equals $\mathcal{B} \equiv \mathcal{A}$.

Non-equivalence is also symmetric: $\mathcal{A} \not\equiv \mathcal{B}$ equals $\mathcal{B} \not\equiv \mathcal{A}$.

Equivalence is transitive: if $\mathcal{A} \equiv \mathcal{B}$ and $\mathcal{B} \equiv \mathcal{C}$, then $\mathcal{A} \equiv \mathcal{C}$ (equal to $\neg\mathcal{A} \equiv \neg\mathcal{C}$); this is very different from "if $\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C}$, then $\mathcal{A} \equiv \mathcal{C}$ and/or $\mathcal{A} \equiv \neg\mathcal{C}$".

Note that we have $(\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C}) \subset ((\mathcal{A} \equiv \mathcal{C}) \cup (\mathcal{A} \equiv \neg\mathcal{C}))$ for $\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C}$ equals a disjunction of conjunctions that is partly included in one equal to $\mathcal{A} \equiv \mathcal{C}$, and partly included in one equal to $\mathcal{A} \equiv \neg\mathcal{C}$, see the disjunctions of conjunctions of topics 10, 12 and 15.

Non-equivalence is also transitive: if $\mathcal{A} \not\equiv \mathcal{B}$ and $\mathcal{B} \not\equiv \mathcal{C}$, then $\mathcal{A} \equiv \mathcal{C}$ and $\neg\mathcal{A} \equiv \neg\mathcal{C}$ (which are equal). After all, $\mathcal{A} \not\equiv \mathcal{B}$ equals both $\mathcal{A} \equiv \neg\mathcal{B}$ and $\neg\mathcal{A} \equiv \mathcal{B}$, while $\mathcal{B} \not\equiv \mathcal{C}$ equals both $\mathcal{B} \equiv \neg\mathcal{C}$ and $\neg\mathcal{B} \equiv \mathcal{C}$.

Equivalence is reflexive: $\mathcal{A} \equiv \mathcal{A}$ is consistent. Also, if $\mathcal{A} \equiv \mathcal{B}$ and $\mathcal{B} \equiv \mathcal{A}$, then $\mathcal{A} \equiv \mathcal{A}$.

Non-equivalence is not reflexive: $\mathcal{A} \not\equiv \mathcal{A}$ is a contradiction (see also topic 11).

## 10. Commutation and Permutation, = vs. ≡

When $\mathcal{A}\ \mathbf{R}\ \mathcal{B}$ equals $\mathcal{B}\ \mathbf{R}\ \mathcal{A}$, the particular **R** is symmetric and/or commutative and permutative.

Symmetry and commutation concern binary statements, while permutation concerns statements with more than two terms. I like to assume that a symmetric relation might be permutative, while a commutative relation is always permutative, so as to distinguish between symmetry and commutation (i.e. I like to assume that commutation is a case of permutation, as pertaining to binary statements).

Take for example $\mathcal{A}\ \mathbf{R}\ \mathcal{B}\ \mathbf{R}\ \mathcal{C}$ (with one kind of **R**). When this statement equals $\mathcal{C}\ \mathbf{R}\ \mathcal{A}\ \mathbf{R}\ \mathcal{B}$, as well as $\mathcal{B}\ \mathbf{R}\ \mathcal{C}\ \mathbf{R}\ \mathcal{A}$, as well as $\mathcal{C}\ \mathbf{R}\ \mathcal{B}\ \mathbf{R}\ \mathcal{A}$, as well as $\mathcal{B}\ \mathbf{R}\ \mathcal{A}\ \mathbf{R}\ \mathcal{C}$, as well as $\mathcal{A}\ \mathbf{R}\ \mathcal{C}\ \mathbf{R}\ \mathcal{B}$, call **R** commutative and permutative.

Of the connectives, disjunction and conjunction are symmetric and commutative and permutative, while implication/inclusion is neither.

### 10.1. Equality

Equality is commutative and permutative.

Take for example $\mathcal{A} = \mathcal{B} = \mathcal{C}$. This statement equals $\mathcal{C} = \mathcal{A} = \mathcal{B}$, as well as $\mathcal{B} = \mathcal{C} = \mathcal{A}$, as well as $\mathcal{C} = \mathcal{B} = \mathcal{A}$, as well as $\mathcal{B} = \mathcal{A} = \mathcal{C}$, as well as $\mathcal{A} = \mathcal{C} = \mathcal{B}$ (for how could it not be equal to these?).

It is unclear whether non-equality is commutative and permutative. It seems that non-equality must at least obey the law of association to be so, see topic 13.

### 10.2. Equivalence

Equivalence is commutative and permutative.

Take for example $\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C}$. Either of the two definitions of equivalence can be used to show that the statement equals $\mathcal{C} \equiv \mathcal{A} \equiv \mathcal{B}$, as well as $\mathcal{B} \equiv \mathcal{C} \equiv \mathcal{A}$, as well as $\mathcal{C} \equiv \mathcal{B} \equiv \mathcal{A}$, as well as $\mathcal{B} \equiv \mathcal{A} \equiv \mathcal{C}$, as well as $\mathcal{A} \equiv \mathcal{C} \equiv \mathcal{B}$. To do so, use the identities named after De Morgan, the law of distribution, and the fact that any contradiction like $\neg\mathcal{A} \cap \mathcal{A}$ denotes nothing.

In addition, equivalence and non-equivalence are commutative and permutative "in a way"; what I mean by this is seen by looking at non-equivalence.

Take for example $\mathcal{A} \not\equiv \mathcal{B} \not\equiv \mathcal{C}$. The statement might say "$\mathcal{A}$ is not equivalent to the non-equivalence of $\mathcal{B}$ and $\mathcal{C}$", or it might say "$\mathcal{C}$ is not equivalent to the non-equivalence of $\mathcal{B}$ and $\mathcal{A}$". In the first case, write $\mathcal{A} \not\equiv (\mathcal{B} \not\equiv \mathcal{C})$; in the second case, write $(\mathcal{A} \not\equiv \mathcal{B}) \not\equiv \mathcal{C}$.

Now $\mathcal{A} \not\equiv (\mathcal{B} \not\equiv \mathcal{C})$ equals both $\mathcal{A} \equiv \neg(\mathcal{B} \not\equiv \mathcal{C})$ and $\neg\mathcal{A} \equiv (\mathcal{B} \not\equiv \mathcal{C})$; of these, the first equals $\mathcal{A} \equiv (\mathcal{B} \equiv \mathcal{C})$, while the second equals both $\neg\mathcal{A} \equiv (\mathcal{B} \equiv \neg\mathcal{C})$ and $\neg\mathcal{A} \equiv (\neg\mathcal{B} \equiv \mathcal{C})$.

Now $(\mathcal{A} \not\equiv \mathcal{B}) \not\equiv \mathcal{C}$ equals both $(\mathcal{A} \not\equiv \mathcal{B}) \equiv \neg\mathcal{C}$ and $\neg(\mathcal{A} \not\equiv \mathcal{B}) \equiv \mathcal{C}$; of these, the first equals both $(\neg\mathcal{A} \equiv \mathcal{B}) \equiv \neg\mathcal{C}$ and $(\mathcal{A} \equiv \neg\mathcal{B}) \equiv \neg\mathcal{C}$, while the second equals $(\mathcal{A} \equiv \mathcal{B}) \equiv \mathcal{C}$.

Because the brackets can be omitted (to see why, use the identities named after De Morgan, the law of distribution, and the fact that any contradiction like $\neg\mathcal{A} \cap \mathcal{A}$ equals $\emptyset$), $\mathcal{A} \not\equiv \mathcal{B} \not\equiv \mathcal{C}$ equals $\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C}$, as well as $\neg\mathcal{A} \equiv \mathcal{B} \equiv \neg\mathcal{C}$, as well as $\neg\mathcal{A} \equiv \neg\mathcal{B} \equiv \mathcal{C}$, as well as $\mathcal{A} \equiv \neg\mathcal{B} \equiv \neg\mathcal{C}$.

Like $\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C}$, each of these statements is commutative and permutative, $\mathcal{A} \not\equiv \mathcal{B} \not\equiv \mathcal{C}$ included. Hence non-equivalence is also commutative and permutative.

In fact, either of the two definitions of equivalence can be used to show that the example equals the disjunction of conjunctions $(\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}) \cup (\neg\mathcal{A} \cap \mathcal{B} \cap \neg\mathcal{C}) \cup (\neg\mathcal{A} \cap \neg\mathcal{B} \cap \mathcal{C}) \cup (\mathcal{A} \cap \neg\mathcal{B} \cap \neg\mathcal{C})$ (again,use the identities named after De Morgan, the law of distribution, and the fact that any contradiction like $\neg\mathcal{A} \cap \mathcal{A}$ equals $\emptyset$).

In fact, any statement, regardless of the number of terms involved, in which equivalence is the only **R**, is equal to some disjunction of conjunctions (and the relations $\cup$ and $\cap$ are always commutative and permutative).

Note that since $\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C}$ equals $\mathcal{A} \not\equiv \mathcal{B} \not\equiv \mathcal{C}$, the negation $\neg(\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C})$ cannot equal $\mathcal{A} \not\equiv \mathcal{B} \not\equiv \mathcal{C}$; instead it equals $\mathcal{A} \equiv (\mathcal{B} \not\equiv \mathcal{C})$, as well as $\mathcal{B} \equiv (\mathcal{C} \not\equiv \mathcal{A})$, as well as $\mathcal{C} \equiv (\mathcal{A} \not\equiv \mathcal{B})$; this can be shown by negating in the usual manner the disjunction of conjunctions given above.

## 11. Tautology, = vs. ≡

Take $\mathcal{A}\ \mathbf{R}\ \mathcal{A}$, and let the relation **R** be either disjunction ($\cup$) or conjunction ($\cap$); then, by the law of tautology, the relation **R** is trivial (a pseudo-relation), and the statement a pseudo-statement equal to $\mathcal{A}$.

Note that since $\mathcal{A} \cup \mathcal{B}$ equals $\neg(\neg\mathcal{A} \cap \neg\mathcal{B})$, the negation of the tautology $\mathcal{A} \cup \mathcal{A}$ equals $\neg\mathcal{A} \cap \neg\mathcal{A}$, by the law of tautology equal to $\neg\mathcal{A}$, as desired.

Note that since $\mathcal{A} \cap \mathcal{B}$ equals $\neg(\neg\mathcal{A} \cup \neg\mathcal{B})$, the negation of the tautology $\mathcal{A} \cap \mathcal{A}$ equals $\neg\mathcal{A} \cup \neg\mathcal{A}$, by the law of tautology equal to $\neg\mathcal{A}$, as desired.

Note that since $\mathcal{A} \cup \mathcal{B}$ equals $\neg\mathcal{A} \subset \mathcal{B}$, the implications/inclusions $\neg\mathcal{A} \subset \mathcal{A}$ and $\mathcal{A} \subset \neg\mathcal{A}$ are tautologies, respectively equal to $\mathcal{A}$ and $\neg\mathcal{A}$.

### 11.1. Equality

Take $\mathcal{A}\ \mathbf{R}\ \mathcal{A}$, and let the relation $\mathbf{R}$ be equality. Since $\mathcal{A} = \mathcal{A}$ simply says that $\mathcal{A}$ denotes the same thing-in-itself as $\mathcal{A}$, it makes sense to consider the $=$ in this statement always a pseudo-relation, and the statement always a pseudo-statement. Accordingly, $\mathcal{A} = \mathcal{A}$ is a tautology.

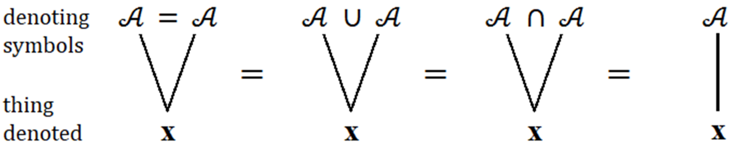


FIGURE 5. tautologies of the symbol $\mathcal{A}$

In fact, consider $\mathcal{A} = \mathcal{A}$ equal to the tautologies $\mathcal{A} \cup \mathcal{A}$ and $\mathcal{A} \cap \mathcal{A}$.

Note that if $\mathcal{A} = \mathcal{A}$ is always a pseudo-statement, then the negation $\neg(\mathcal{A} = \mathcal{A})$ is always equal to $\neg\mathcal{A}$ (instead of to the contradiction $\mathcal{A} \neq \mathcal{A}$).

Further note that the equality $\mathcal{A} = \mathcal{B}$ can be put equal to $\mathcal{A} = (\mathcal{B} = \mathcal{A})$ and might therefore be replaced by the always trivial $\mathcal{A} = \mathcal{A}$ (also recall how we stated the law of reflexivity). However, if $\mathcal{A} = \mathcal{A}$ is always trivial, it must somehow be different from $\mathcal{A} = \mathcal{B}$ (which, with respect to the symbols that denote, behaves like a proper relation). A similar difficulty results from replacing $\mathcal{A} \cup \mathcal{A}$ and $\mathcal{A} \cap \mathcal{A}$ by respectively $\mathcal{A} \cup \mathcal{B}$ and $\mathcal{A} \cap \mathcal{B}$ when $\mathcal{A} = \mathcal{B}$. In this light, recall figure 4.

### 11.2. Equivalence

Take $\mathcal{A}\ \mathbf{R}\ \mathcal{A}$, and let the relation $\mathbf{R}$ be implication/inclusion. Since $\mathcal{A} \subset \mathcal{A}$ equals $\neg\mathcal{A} \cup \mathcal{A}$, it does not obey the law of tautology.

Now $\mathcal{A} \equiv \mathcal{A}$ equals $(\mathcal{A} \subset \mathcal{A}) \cap (\mathcal{A} \supset \mathcal{A})$ by the first definition of equivalence, and $(\mathcal{A} \cup \mathcal{A}) \subset (\mathcal{A} \cap \mathcal{A})$ by the second definition of equivalence; both statements equal $\mathcal{A} \subset \mathcal{A}$ by the law of tautology.

Take $\mathcal{A}\ \mathbf{R}\ \mathcal{A}$, and let the relation $\mathbf{R}$ be equivalence. Since $\mathcal{A} \equiv \mathcal{A}$ equals $\mathcal{A} \subset \mathcal{A}$, it does not obey the law of tautology.

Note that because $\mathcal{A} \equiv \mathcal{A}$ equals $\mathcal{A} \cup \neg\mathcal{A}$, the negation $\mathcal{A} \not\equiv \mathcal{A}$ equals the contradiction $\neg\mathcal{A} \cap \mathcal{A}$.

Here, equivalence and non-equivalence correspond to some connection from one thing-in-itself to some other through only one connective, as if they are basic relations instead of non-basic relations.

## 12. Distribution, $=$ vs. $\equiv$

For sure, the law of distribution applies to any conjunction of disjunctions; in a sense it interchanges $\cap$ and $\cup$. By the law of distribution $\mathcal{A} \cap (\mathcal{B} \cup \mathcal{C})$ is equal to $(\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{B} \cap \mathcal{C})$.

### 12.1. Equality

Unless the statement $\mathcal{A} = \mathcal{B}$ is a pseudo-statement equal to a single non-basic term, and this non-basic term equals some conjunction of disjunctions, the law of distribution, as given above, does not apply.

### 12.2. Equivalence

By either definition of equivalence, the statement $\mathcal{A} \equiv \mathcal{B}$ equals a conjunction of disjunctions, so that the law of distribution applies.

By the first definition of equivalence, $\mathcal{A} \equiv \mathcal{B}$ equals $(\mathcal{A} \subset \mathcal{B}) \cap (\mathcal{A} \supset \mathcal{B})$, equal to $(\neg\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup \neg\mathcal{B})$, and by the law of distribution this is equal to $(\neg\mathcal{A} \cap \neg\mathcal{B}) \cup (\mathcal{B} \cap \mathcal{A})$ (since contradictions like $\neg\mathcal{A} \cap \mathcal{A}$ denote nothing, and can be omitted).

By the second definition, $\mathcal{A} \equiv \mathcal{B}$ equals $(\mathcal{A} \cup \mathcal{B}) \subset (\mathcal{A} \cap \mathcal{B})$, equal to $(\neg\mathcal{A} \cap \neg\mathcal{B}) \cup (\mathcal{A} \cap \mathcal{B})$, equal to $\neg((\mathcal{A} \cup \mathcal{B}) \cap (\neg\mathcal{A} \cup \neg\mathcal{B}))$, and by the law of distribution this is equal to $\neg((\mathcal{A} \cap \neg\mathcal{B}) \cup (\mathcal{B} \cap \neg\mathcal{A}))$ (since contradictions like $\neg\mathcal{A} \cap \mathcal{A}$ denote nothing, and can be omitted).

Moreover, by the law of distribution, the first and second definition of equivalence are equal.

Indeed, by the first definition and the law of distribution, $\mathcal{A} \equiv \mathcal{B}$ is equal to $(\neg\mathcal{A} \cap \neg\mathcal{B}) \cup (\mathcal{B} \cap \mathcal{A})$, equal to $\neg(\neg\mathcal{A} \cap \neg\mathcal{B}) \subset (\mathcal{B} \cap \mathcal{A})$, equal to $(\mathcal{A} \cup \mathcal{B}) \subset (\mathcal{A} \cap \mathcal{B})$, which is the second definition of $\mathcal{A} \equiv \mathcal{B}$.

Hence, given some disjunction of conjunctions $(\neg\mathcal{A} \cap \neg\mathcal{B}) \cup (\mathcal{B} \cap \mathcal{A})$, equal to some equivalence $\mathcal{A} \equiv \mathcal{B}$ (also recall the computational table of this equivalence), the second definition of equivalence can be directly obtained by applying the identities named after De Morgan. On the other hand, the first definition is obtained by adding two contradictions (namely $\neg\mathcal{A} \cap \mathcal{A}$ and $\neg\mathcal{B} \cap \mathcal{B}$, both equal to $\emptyset$) to the disjunction of conjunctions, and by then applying the law of distribution in the reverse, and by then applying the identities named after De Morgan.

## 13. Association, $=$ vs. $\equiv$

By the law of association, any number of terms, in their given order, and including any relations in between, of some "unambiguous" statement, may be bracketed without harm (as long as the brackets do not interfere with each other or any existing ones).

That is, when unambiguous, $\mathcal{A}\ \mathbf{R}\ \mathcal{B}$ not only equals $(\mathcal{A}\ \mathbf{R}\ \mathcal{B})$; it also equals $\mathcal{A}\ \mathbf{R}\ (\mathcal{B})$, as well as $(\mathcal{A})\ \mathbf{R}\ \mathcal{B}$, as well as $(\mathcal{A})\ \mathbf{R}\ (\mathcal{B})$ (where $\mathcal{A}$ and/or $\mathcal{B}$ may be non-basic terms, equal to some proper statement).

When brackets would affect the statement, the statement is called "ambiguous" and does not obey the law of association. For example, $\mathcal{A} \cap \mathcal{B} \cup \mathcal{C}$ is ambiguous because $\mathcal{A} \cap (\mathcal{B} \cup \mathcal{C})$ and $(\mathcal{A} \cap \mathcal{B}) \cup \mathcal{C}$ are different; on the other hand, the statements $\mathcal{A} \cap (\mathcal{B} \cup \mathcal{C})$ and $(\mathcal{A} \cap \mathcal{B}) \cup \mathcal{C}$ are unambiguous.

One reason to assign brackets to some statement that is unambiguous is to rewrite the statement as some different combination of symbols with identical meaning.

For example, use the identity $(\mathcal{A} \cup \mathcal{B}) = (\neg\mathcal{A} \subset \mathcal{B})$ to rewrite the statement $\mathcal{C} \cup \mathcal{D} \cup \mathcal{E}$. One approach is to bracket $\mathcal{C}$ and $\mathcal{D}$ and to apply the identity to the statement in brackets, followed by an application to the disjunction that remains. The result is $\neg(\neg\mathcal{C} \subset \mathcal{D}) \subset \mathcal{E}$ (compare this to e.g. $\neg\mathcal{C} \subset \neg\mathcal{D} \subset \mathcal{E}$, which is ambiguous).

Another reason to assign brackets to some statement that is unambiguous is to negate (parts of) that statement.

For example, let $\mathcal{A}\ \mathbf{R}\ \mathcal{B}$ be unambiguous; then $\neg\mathcal{A}\ \mathbf{R}\ \mathcal{B}$ is ambiguous, for it might equal $\neg(\mathcal{A}\ \mathbf{R}\ \mathcal{B})$, as well as $\neg(\mathcal{A})\ \mathbf{R}\ \mathcal{B}$, and these are different.

Note that instead of the brackets ( and ), one could use any other pair of symbols or even different degrees of spacing to make associations. Also context may provide implicit bracketing, so that context suffices to make some ambiguous statement unambiguous.

### 13.1. Equality

Clearly, equality obeys the law of association.

Even when for example $\mathcal{A} = (\mathcal{B} = \mathcal{C})$ means that $\mathcal{A}$ stands for $\mathcal{B} = \mathcal{C}$, and $(\mathcal{A} = \mathcal{B}) = \mathcal{C}$ means that $\mathcal{C}$ stands for $\mathcal{A} = \mathcal{B}$, the statement $\mathcal{A} = \mathcal{B} = \mathcal{C}$ has an unambiguous interpretation.

It is unclear whether non-equality obeys the law of association.

For example, let $\mathcal{A}$ equal the noun "pipe", let $\mathcal{B}$ equal the noun "apple", and let $\mathcal{C}$ equal the noun "moustache"; then $\mathcal{A} \neq (\mathcal{B} \neq \mathcal{C})$ says "a pipe and the statement that an apple is not a moustache are not equal", while $(\mathcal{A} \neq \mathcal{B}) \neq \mathcal{C}$ says "a moustache and the statement that a pipe is not an apple are not equal"; meanwhile $\mathcal{A} \neq \mathcal{B} \neq \mathcal{C}$ says "a pipe is not equal to an apple that is not equal to a moustache". Are these three statements equal (i.e. do they correspond to the same situation)?

Note that the law of association is not obeyed by for example $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$; after all, $\mathcal{A} = (\mathcal{B} \cup \mathcal{C})$ and $(\mathcal{A} = \mathcal{B}) \cup \mathcal{C}$ are different.

### 13.2. Equivalence

Equivalence obeys the law of association.

Either definition of equivalence can be used to show that for example $\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C}$ is equal to the same disjunction of conjunctions as $\mathcal{A} \equiv (\mathcal{B} \equiv \mathcal{C})$ and $(\mathcal{A} \equiv \mathcal{B}) \equiv \mathcal{C}$, see topic 10.

When either definition of equivalence is used to rewrite a statement like $\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C}$, brackets are required. Without brackets, the first definition cannot be applied and the second might give $(\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}) \subset (\mathcal{A} \cap \mathcal{B} \cap \mathcal{C})$, which is not identical to $\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C}$.

Brackets are also required to rewrite any equivalence as some non-equivalence, or vice versa, by means of either definition; for example, to rewrite $\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C}$ as $\mathcal{A} \not\equiv \mathcal{B} \not\equiv \mathcal{C}$, or vice versa, by means of either definition.

Non-equivalence also obeys the law of association. Indeed, either definition of equivalence can be used to show that for example $\mathcal{A} \not\equiv \mathcal{B} \not\equiv \mathcal{C}$ is equal to the same disjunction of conjunctions as $\mathcal{A} \not\equiv (\mathcal{B} \not\equiv \mathcal{C})$ and $(\mathcal{A} \not\equiv \mathcal{B}) \not\equiv \mathcal{C}$, see topic 10.

Note that the law of association is not obeyed by for example $\mathcal{A} \equiv \mathcal{B} \cup \mathcal{C}$; after all, $\mathcal{A} \equiv (\mathcal{B} \cup \mathcal{C})$ and $(\mathcal{A} \equiv \mathcal{B}) \cup \mathcal{C}$ are different.

## 14. Propositional Meaning, $=$ vs. $\equiv$

Traditional (Aristotelian) logic allows for four kinds of statement, namely the universal affirmative A, the universal negative E, the particular affirmative I, and the particular negative O; any of these propositions counts two cases. We have (according to Leibniz and Boole, more or less, and De Morgan [5, 2, 7])

| | | | | |
|---|---|---|---|---|
| $\mathrm{A}^1$ | $=$ | all $\mathcal{A}$s are $\mathcal{B}$s | $=$ | $\mathcal{A} \subset \mathcal{B}$ |
| $\mathrm{A}_2$ | $=$ | all $\mathcal{B}$s are $\mathcal{A}$s | $=$ | $\mathcal{A} \supset \mathcal{B}$ |
| $\mathrm{E}^1$ | $=$ | no $\mathcal{A}$s are $\mathcal{B}$s | $=$ | $\mathcal{A} \subset \neg\mathcal{B}$ |
| $\mathrm{E}_2$ | $=$ | no $\neg\mathcal{A}$s are $\neg\mathcal{B}$s | $=$ | $\neg\mathcal{A} \subset \mathcal{B}$ |
| $\mathrm{I}^1$ | $=$ | some $\mathcal{A}$s are $\mathcal{B}$s | $=$ | $\mathcal{A} \cap \mathcal{B}$ |
| $\mathrm{I}_2$ | $=$ | some $\neg\mathcal{A}$s are $\neg\mathcal{B}$s | $=$ | $\neg\mathcal{A} \cap \neg\mathcal{B}$ |
| $\mathrm{O}^1$ | $=$ | some $\mathcal{A}$s are not $\mathcal{B}$s | $=$ | $\mathcal{A} \cap \neg\mathcal{B}$ |
| $\mathrm{O}_2$ | $=$ | some $\mathcal{B}$s are not $\mathcal{A}$s | $=$ | $\neg\mathcal{A} \cap \mathcal{B}$ |

I intend to provide a more thorough investigation of the connectives in terms of the quantifiers “all” and “some” and the equality-verb “to be” elsewhere (it might help to answer the question raised with topic 8); see also table 2 and appendix C.

TABLE 2. relations of logic, quantifiers and "to be": an interpretation

| required for | $\cup$ | $\subset$ | $\supset$ | $\cap$ | $\equiv$ | $=$ |
|---|---|---|---|---|---|---|
| *all are all* | *no* | *no* | *no* | *no* | *no* | *yes* |
| *all are some* | *no* | *yes* | *no* | *no* | *yes* | *no* |
| *some are all* | *no* | *no* | *yes* | *no* | *yes* | *no* |
| *some are some* | *no* | *no* | *no* | *yes* | *no* | *no* |

let any term have "some" part $\emptyset$

### 14.1. Equality

In $A^1$ and $A_2$, substitute $\neg\mathcal{A}$ for $\mathcal{B}$ to obtain

$$A^1 = \text{all } \mathcal{A}\text{s are } \neg\mathcal{A}\text{s} = \neg\mathcal{A} \cup \neg\mathcal{A}$$
$$A_2 = \text{all } \neg\mathcal{A}\text{s are } \mathcal{A}\text{s} = \mathcal{A} \cup \mathcal{A}$$

If $(\neg\mathcal{A} \cup \neg\mathcal{A}) = (\neg\mathcal{A} = \neg\mathcal{A})$ and $(\mathcal{A} \cup \mathcal{A}) = (\mathcal{A} = \mathcal{A})$, then the reflexive tautology $\neg\mathcal{A} = \neg\mathcal{A}$ says "all $\mathcal{A}$s are $\neg\mathcal{A}$s" (i.e. "no $\mathcal{A}$s are $\mathcal{A}$s"), while the reflexive tautology $\mathcal{A} = \mathcal{A}$ says "all $\neg\mathcal{A}$s are $\mathcal{A}$s" (i.e. "no $\neg\mathcal{A}$s are $\neg\mathcal{A}$s").

In $E^1$ and $E_2$, substitute $\mathcal{A}$ for $\mathcal{B}$ to obtain identical statements.

In $I^1$ and $I_2$, substitute $\mathcal{A}$ for $\mathcal{B}$ to obtain

$$I^1 = \text{some } \mathcal{A}\text{s are } \mathcal{A}\text{s} = \mathcal{A} \cap \mathcal{A}$$
$$I_2 = \text{some } \neg\mathcal{A}\text{s are } \neg\mathcal{A}\text{s} = \neg\mathcal{A} \cap \neg\mathcal{A}$$

If $(\mathcal{A} \cap \mathcal{A}) = (\mathcal{A} = \mathcal{A})$ and $(\neg\mathcal{A} \cap \neg\mathcal{A}) = (\neg\mathcal{A} = \neg\mathcal{A})$, then the reflexive tautology $\mathcal{A} = \mathcal{A}$ says "some $\mathcal{A}$s are $\mathcal{A}$s" (i.e. "some $\mathcal{A}$s are not $\neg\mathcal{A}$s"), while the reflexive tautology $\neg\mathcal{A} = \neg\mathcal{A}$ says "some $\neg\mathcal{A}$s are $\neg\mathcal{A}$s" (i.e. "some $\neg\mathcal{A}$s are not $\mathcal{A}$s").

In $O^1$ and $O_2$, substitute $\neg\mathcal{A}$ for $\mathcal{B}$ to obtain identical statements.

With these substitutions in place, $\mathcal{A} = (A_2 = E_2 = I^1 = O^1)$, while $\neg\mathcal{A} = (A^1 = E^1 = I_2 = O_2)$.

Note that if $(\mathcal{A} = \mathcal{A}) = (\mathcal{A} = (\mathcal{A} = \mathcal{B})) = (\mathcal{A} = \mathcal{B})$, the same propositions should hold with respect to $\mathcal{A}$ and $\mathcal{B}$.

### 14.2. Equivalence

In $E^1$ and $E_2$, substitute $\neg\mathcal{A}$ for $\mathcal{B}$ to obtain

$$E^1 = \text{no } \mathcal{A}\text{s are } \neg\mathcal{A}\text{s} = \mathcal{A} \subset \mathcal{A}$$
$$E_2 = \text{no } \neg\mathcal{A}\text{s are } \mathcal{A}\text{s} = \neg\mathcal{A} \subset \neg\mathcal{A}$$

Since $(\mathcal{A} \subset \mathcal{A}) = (\mathcal{A} \equiv \mathcal{A})$, the reflexive $\mathcal{A} \equiv \mathcal{A}$ says "no $\mathcal{A}$s are $\neg\mathcal{A}$s" (i.e. "all $\mathcal{A}$s are $\mathcal{A}$s" ); since $(\neg\mathcal{A} \subset \neg\mathcal{A}) = (\neg\mathcal{A} \equiv \neg\mathcal{A})$, the reflexive

$\neg\mathcal{A} \equiv \neg\mathcal{A}$ says "no $\neg\mathcal{A}$s are $\mathcal{A}$s" (i.e. "all $\neg\mathcal{A}$s are $\neg\mathcal{A}$s"). These two statements are equal (they correspond to the same situation of things-in-themselves).

In $A^1$ and $A_2$, substitute $\mathcal{A}$ for $\mathcal{B}$ to obtain a statement identical to $E^1$.

Note that when $\neg\mathcal{A}$ is substituted for $\mathcal{B}$ in $I^1$ and $I_2$, or $\mathcal{A}$ is substituted for $\mathcal{B}$ in $O^1$ and $O_2$, then the contradiction $(\neg\mathcal{A} \cap \mathcal{A}) = (\mathcal{A} \not\equiv \mathcal{A})$ is obtained, which says "some $\mathcal{A}$s are not $\mathcal{A}$s".

Furthermore, the propositional meaning of the first definition of equivalence $(\mathcal{A} \subset \mathcal{B}) \cap (\mathcal{A} \supset \mathcal{B})$ is "some "all $\mathcal{A}$s are $\mathcal{B}$s" are "all $\mathcal{B}$s are $\mathcal{A}$s"", as well as "some "all $\neg\mathcal{B}$s are $\neg\mathcal{A}$s" are "all $\neg\mathcal{A}$s are $\neg\mathcal{B}$s"".

Furthermore, the propositional meaning of the second definition of equivalence $(\mathcal{A} \cup \mathcal{B}) \subset (\mathcal{A} \cap \mathcal{B})$ is "all "all $\neg\mathcal{A}$s are $\mathcal{B}$s" are "some $\mathcal{A}$s are $\mathcal{B}$s"", as well as "all "all $\neg\mathcal{B}$s are $\mathcal{A}$s" are "some $\mathcal{B}$s are $\mathcal{A}$s"", as well as "all "all $\mathcal{A}$s are $\neg\mathcal{B}$s" are "some $\neg\mathcal{A}$s are $\neg\mathcal{B}$s"", as well as "all "all $\mathcal{B}$s are $\neg\mathcal{A}$s" are "some $\neg\mathcal{B}$s are $\neg\mathcal{A}$s"".

## 15. Genesis, = vs. ≡

By "genesis" I mean the obtainment of statements from the negation of the empty term $\emptyset$, using the laws of logic, and given some number of terms.

Now the negation of the empty term $\emptyset$ equals the universal term **U**.

Now, the universal term **U** is equal to as many disjunctions of some term and the negation of that term, as there are terms given. For example, let there be given the terms $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, ... (imagine that up to an infinite number of letters exist); then

$$\begin{array}{lllllllll} \mathbf{U} & = & \mathcal{A} \cup \neg\mathcal{A} & = & \mathcal{B} \cup \neg\mathcal{B} & = & \mathcal{C} \cup \neg\mathcal{C} & = & \dots \\ & = & \mathcal{A} \equiv \mathcal{A} & = & \mathcal{B} \equiv \mathcal{B} & = & \mathcal{C} \equiv \mathcal{C} & = & \dots \\ & = & \neg\mathcal{A} \equiv \neg\mathcal{A} & = & \neg\mathcal{B} \equiv \neg\mathcal{B} & = & \neg\mathcal{C} \equiv \neg\mathcal{C} & = & \dots \end{array}$$

Note that when these terms are combined into statements (that equal non-basic terms), the universal term **U** will also equal as many disjunctions of some statement and the negation of that statement, as there are non-basic terms (for example, **U** will also equal $(\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C} \equiv \mathcal{D}) \cup \neg(\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C} \equiv \mathcal{D})$; see also [14]).

Of course, any tautology of the universal term **U** is equal to **U**. That is

$$\begin{array}{lllllll} \mathbf{U} & = & \mathbf{U} = \mathbf{U} & = & \mathbf{U} = \mathbf{U} = \mathbf{U} & = & \dots \\ & = & \mathbf{U} \cup \mathbf{U} & = & \mathbf{U} \cup \mathbf{U} \cup \mathbf{U} & = & \dots \\ & = & \mathbf{U} \cap \mathbf{U} & = & \mathbf{U} \cap \mathbf{U} \cap \mathbf{U} & = & \dots \end{array}$$

Of the tautologies of **U**, the conjunctions obey the law of distribution. For example

$$\begin{aligned}
\mathbf{U} &= \mathbf{U} \cap \mathbf{U} \\
&= (\mathcal{A} \cup \neg\mathcal{A}) \cap (\mathcal{A} \cup \neg\mathcal{A}) \\
&= (\mathcal{A} \cap \mathcal{A}) \cup (\mathcal{A} \cap \neg\mathcal{A}) \cup (\neg\mathcal{A} \cap \mathcal{A}) \cup (\neg\mathcal{A} \cap \neg\mathcal{A}) \\
&= (\mathcal{A} \cup \neg\mathcal{A}) = (\mathcal{A} \equiv \mathcal{A}) = (\neg\mathcal{A} \equiv \neg\mathcal{A})
\end{aligned}$$

For example

$$\begin{aligned}
\mathbf{U} &= \mathbf{U} \cap \mathbf{U} \\
&= (\mathcal{A} \cup \neg\mathcal{A}) \cap (\mathcal{B} \cup \neg\mathcal{B}) \\
&= (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \neg\mathcal{B}) \cup (\neg\mathcal{A} \cap \mathcal{B}) \cup (\neg\mathcal{A} \cap \neg\mathcal{B}) \\
&= (\mathcal{A} \equiv \mathcal{B}) \cup \neg(\mathcal{A} \equiv \mathcal{B})
\end{aligned}$$

For example

$$\begin{aligned}
\mathbf{U} &= \mathbf{U} \cap \mathbf{U} \cap \mathbf{U} \\
&= (\mathcal{A} \cup \neg\mathcal{A}) \cap (\mathcal{B} \cup \neg\mathcal{B}) \cap (\mathcal{C} \cup \neg\mathcal{C}) \\
&= (\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}) \cup (\mathcal{A} \cap \neg\mathcal{B} \cap \mathcal{C}) \cup (\neg\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}) \cup \\
&\quad (\neg\mathcal{A} \cap \neg\mathcal{B} \cap \mathcal{C}) \cup (\mathcal{A} \cap \mathcal{B} \cap \neg\mathcal{C}) \cup (\mathcal{A} \cap \neg\mathcal{B} \cap \neg\mathcal{C}) \cup \\
&\quad (\neg\mathcal{A} \cap \mathcal{B} \cap \neg\mathcal{C}) \cup (\neg\mathcal{A} \cap \neg\mathcal{B} \cap \neg\mathcal{C}) \\
&= (\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C}) \cup \neg(\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C})
\end{aligned}$$

And so on.

Hence given any two terms and their negations, the universal term **U** will equal the disjunction of all different conjunctions (four in number) of these terms, equal to the disjunction of all different equivalences (two in number) of these terms.

Likewise, given any three terms and their negations, the universal term **U** will equal the disjunction of all different conjunctions (eight in number) of these terms, equal to the disjunction of all different equivalences (again, two in number) of these terms.

And so on.

In addition, let there be some variable statement $\mathcal{S}$ (equal to $\mathcal{S} \equiv \mathbf{U}$, equal to $\neg\mathcal{S} \equiv \emptyset$) which at different places and/or times can equal different particular statements in **U**, while its negation equals everything else. Then $\mathcal{S}$ can be used to select some particular statement over others, for example $\mathcal{A} \cap \mathcal{B}$ over $\neg\mathcal{A} \cup \neg\mathcal{B}$, or for example $\mathcal{A} \equiv \mathcal{B}$ over $\neg(\mathcal{A} \equiv \mathcal{B})$.

Finally note that any single term may be "split into" two disjoint conjunctions of two terms. For example $\mathcal{A} = (\mathcal{A} \cap \mathbf{U}) = (\mathcal{A} \cap (\mathcal{B} \cup \neg\mathcal{B})) = ((\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \neg\mathcal{B}))$. Likewise, every conjunction of two terms can be split into two disjoint conjunctions of three terms. And so on. Hence in **U**, the single basic and/or non-basic terms include conjunctions of two terms, and these include conjunctions of three terms, and so on.

I intend to elaborate upon the "layered structure" of disjoint conjunctions, equal to our universe **U**, elsewhere. For now, the short outline presented here must do.

### 15.1. Equality

"Genesis" cannot be used to obtain statements with equality. Instead, equality can be used for genesis.

Indeed, equality (as $=_{df}$) can be used to state that $\neg\emptyset = \mathbf{U}$, and that $\mathbf{U} = (\mathbf{U} \cap \mathbf{U})$, and so on.

Indeed, equality (as $=_v$) can be used to state, for example, that $\mathcal{S} = (\mathcal{A} \equiv \mathcal{B})$, while $\neg\mathcal{S} = \neg(\mathcal{A} \equiv \mathcal{B})$.

Note that the universal term **U** simply is what it is. What I call genesis of statements from **U** (with aid of $=_{df}$) merely helps us to perceive **U** more clearly; in other words, it is a genesis of perceptions (of that which already exists; see also [6]).

Note that to select particular statements from **U** (with aid of e.g. $=_v$), while ignoring others, might require some form of "measurement" or "choice". Whatever value the variable $\mathcal{S}$ takes on, it does not simply equal that value by being itself (for by itself it is a variable): some form of agency seems required to force it to take on that value (for example, by means of $=_v$). In fact, recall the "breaking of symmetry" of topic 7, and let $\parallel$ equal "measure/choose between"; then we could state, for example

$$\begin{aligned}
\mathbf{U} &= (\mathcal{S} \neq \mathcal{A}) \cup ((\mathcal{S} = \mathcal{A}) \parallel (\neg\mathcal{S} = \neg\mathcal{A})) \\
&= (\neg\mathcal{S} \neq \neg\mathcal{A}) \cup ((\mathcal{S} = \mathcal{A}) \parallel (\neg\mathcal{S} = \neg\mathcal{A})) \\
&= ((\mathcal{S} = \neg\mathcal{A}) \parallel (\neg\mathcal{S} = \mathcal{A})) \cup (\mathcal{S} \neq \neg\mathcal{A}) \\
&= ((\mathcal{S} = \neg\mathcal{A}) \parallel (\neg\mathcal{S} = \mathcal{A})) \cup (\neg\mathcal{S} \neq \mathcal{A})
\end{aligned}$$

### 15.2. Equivalence

"Genesis" can be used to obtain all possible equivalences.

For example, **U** can be shown equal to $\mathcal{A} \equiv \mathcal{B}$ and/or $\neg(\mathcal{A} \equiv \mathcal{B})$; note that each of these equivalences equals a disjunction of two conjunctions.

For example, **U** can be shown equal to $\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C}$ and/or $\neg(\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C})$; note that each of these equivalences equals a disjunction of four conjunctions.

Likewise, it should be possible to split **U** into $\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C} \equiv \mathcal{D}$ and $\neg(\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C} \equiv \mathcal{D})$, and each of these equivalences should equal a disjunction of eight conjunctions.

And so on.

Finally, note that $\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C}$ is fully included in $(\mathcal{A} \equiv \mathcal{B}) \cup \neg(\mathcal{A} \equiv \mathcal{B})$, equal to $(\mathcal{B} \equiv \mathcal{C}) \cup \neg(\mathcal{B} \equiv \mathcal{C})$, equal to $(\mathcal{C} \equiv \mathcal{A}) \cup \neg(\mathcal{C} \equiv \mathcal{A})$.

Likewise, $\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C} \equiv \mathcal{D}$ is fully included in $(\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C}) \cup \neg(\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C})$, equal to etc..

And so on.

Given these equivalences, use $\mathcal{S}$ to select one equivalence over another.

Note that $(\mathbf{U} \equiv \mathcal{S}) = (\emptyset \equiv \neg\mathcal{S}) = \mathcal{S}$, and that $(\mathbf{U} \equiv \neg\mathcal{S}) = (\emptyset \equiv \mathcal{S}) = \neg\mathcal{S}$, so that $\mathbf{U} = ((\mathbf{U} \equiv \mathcal{S}) \cup (\mathbf{U} \equiv \neg\mathcal{S})) = ((\emptyset \equiv \neg\mathcal{S}) \cup (\emptyset \equiv \mathcal{S}))$.

Finally, note that $(\mathbf{U} \equiv \mathbf{U}) = (\emptyset \equiv \emptyset) = \mathbf{U}$, and that $(\mathbf{U} \equiv \emptyset) = (\emptyset \equiv \mathbf{U}) = \emptyset$; so these equivalences can also be obtained with genesis.

## Conclusion

Topic by topic, the presumed differences between $=$ and $\equiv$ are

### 1.0. Definition

$=$ is not defined by other relations (unless as the negation of $\neq$); as such, it is a "basic" relation.

$\equiv$ is defined by other relations, for $\mathcal{A} \equiv \mathcal{B}$ equals $(\mathcal{A} \subset \mathcal{B}) \cap (\mathcal{A} \supset \mathcal{B})$ (the first definition of equivalence), as well as $(\mathcal{A} \cup \mathcal{B}) \subset (\mathcal{A} \cap \mathcal{B})$ (the second definition of equivalence); it is a "non-basic" relation. Each of the two definitions has sixty-four equal forms by the identities named after De Morgan. Also, $\mathcal{A} \not\equiv \mathcal{B}$ equals $\neg(\mathcal{A} \equiv \mathcal{B})$.

### 2.0. Linguistics

$=$ does not equal any linguistic connective (like $\cup$ and $\cap$ do) or preposition (like $\subset$ does); however, it does equal a verb ("to equal").

$\equiv$ (by its two definitions) equals two kinds of combinations of connectives and prepositions, as well as a verb-phrase ("to be equivalent to").

### 3.0. Computational Gates and Tables

$\mathcal{A} = \mathcal{B}$ is not implemented by some computational gate that counts two different inputs and one output; instead it is implemented by a buffer-gate or a wire. $\mathcal{A} = \mathcal{B}$ does not equal a computational table of two terms.

$\mathcal{A} \equiv \mathcal{B}$ (by its two definitions) can be implemented by one and another kind of combination of computational gates that count two inputs and one output (i.e. gates that represent connectives). $\mathcal{A} \equiv \mathcal{B}$ equals a computational table of two terms.

### 4.0. Denotation

$=$ does not "correspond" to some "connection" from one "thing-in-itself" to some other. When $=$ is placed on the same footing as some relation that does correspond to some connection from one thing-in-itself to some other, it may be treated as a a "pseudo-relation" and $\mathcal{A} = \mathcal{B}$ may be treated as a "pseudo-statement", equal to a single term $\mathcal{A}$ (as well as $\mathcal{B}$).

$\equiv$ (by its two definitions) indirectly corresponds to some connection(s) from one thing-in-itself to some other.

## 5.0. Application

$=$ (as $=_v$) relates variables to values, or (as $=_{df}$) definienda to their definitions.

$\equiv$ (as $\equiv_{\mathfrak{p}}$)may relate terms that have some "property" $\mathfrak{p}$ in common (perhaps call equivalence "equality with respect to property").

## 6.0. Negation of Terms

$\mathcal{A} = \mathcal{B}$, $\mathcal{A} = \neg\mathcal{B}$, $\neg\mathcal{A} = \mathcal{B}$, and $\neg\mathcal{A} = \neg\mathcal{B}$ are non-equal to one another, for these statements may "reduce" to single terms that cannot be equal.

$\mathcal{A} \equiv \mathcal{B}$ equals $\neg\mathcal{A} \equiv \neg\mathcal{B}$ by definition, while $\mathcal{A} \equiv \neg\mathcal{B}$ equals $\neg\mathcal{A} \equiv \mathcal{B}$ by definition.

## 7.0. Negation of the Relation

$\neg =$ may equal $\neq$ when $=$ is not treated as a pseudo-relation; however, when $=$ is treated as a pseudo-relation, the statement $\neg(\mathcal{A} = \mathcal{B})$ equals the single term $\neg\mathcal{A}$, as well as the single term $\neg\mathcal{B}$. Next, non-equality purely concerns symbols that denote (like $=$ does) and/or corresponds to some connection between things-in-themselves (like any "proper" relation does). Moreover, if $\mathcal{A} \neq \mathcal{B}$ is a contradiction, equal to $\emptyset$, and $\mathcal{A} = \mathcal{B}$ would be the negation of $\mathcal{A} \neq \mathcal{B}$, then $\mathcal{A} = \mathcal{B}$ would be $\mathbf{U} = ((\mathcal{A} = \mathcal{B}) \cup (\neg\mathcal{A} = \neg\mathcal{B}))$, which is not that equality (unless $\neg\mathcal{A} = \neg\mathcal{B}$ equals $\emptyset$). In this sense, if a contradictive equality yields non-equality, there might be no way back to that equality, would the non-equality become contradictive: I call this a case of "symmetry breaking". Moreover, if $\mathcal{A} = \mathcal{B}$ is a contradiction, equal to $\emptyset$, and $\mathcal{A} \neq \mathcal{B}$ would be the negation of $\mathcal{A} = \mathcal{B}$, then $\mathcal{A} \neq \mathcal{B}$ would be $\mathcal{A} \cup \neg\mathcal{A}$, which means that, if the terms $\mathcal{A}$ and $\mathcal{B}$ are only partly non-equal, part or all of $\neg\mathcal{A}$ has to be $\emptyset$.

$\neg \equiv$ is put equal to $\not\equiv$, so that $\neg(\mathcal{A} \equiv \mathcal{B})$ is equal to $\mathcal{A} \not\equiv \mathcal{B}$, and so that $\neg(\mathcal{A} \equiv \neg\mathcal{B})$ is equal to $\mathcal{A} \not\equiv \neg\mathcal{B}$. Consequently, $\mathcal{A} \not\equiv \mathcal{B}$ is equal to $\mathcal{A} \equiv \neg\mathcal{B}$, while $\mathcal{A} \not\equiv \neg\mathcal{B}$ is equal to $\mathcal{A} \equiv \mathcal{B}$. Interestingly, the negation of the first definition (called a statement of the "$\cap$-kind") of $\mathcal{A} \equiv \mathcal{B}$, equals the second definition (called a statement of the "$\cup$-kind") of $\mathcal{A} \not\equiv \mathcal{B}$, and vice versa. Likewise, the negation of the second definition (called a statement of the "$\cup$-kind") of $\mathcal{A} \equiv \mathcal{B}$, equals the first definition (called a statement of the "$\cap$–kind") of $\mathcal{A} \not\equiv \mathcal{B}$, and vice versa.

Note that the negation of $\mathcal{A} = \mathcal{B} = \mathcal{C}$ (if not a pseudo-statement) might equal either $\mathcal{A} = \mathcal{B} \neq \mathcal{C}$ or $\mathcal{A} \neq \mathcal{B} = \mathcal{C}$ or $\mathcal{C} \neq \mathcal{A} = \mathcal{B}$ or $\mathcal{A} \neq \mathcal{B} \neq \mathcal{C}$. Note that the negation of $\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C}$ is equal to some statement with only one $\not\equiv$ (namely $\mathcal{A} \equiv \mathcal{B} \not\equiv \mathcal{C}$ and $\mathcal{A} \not\equiv \mathcal{B} \equiv \mathcal{C}$ and $\mathcal{C} \not\equiv \mathcal{A} \equiv \mathcal{B}$), for $\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C}$ and $\mathcal{A} \not\equiv \mathcal{B} \not\equiv \mathcal{C}$ are equal.

### 8.0. Relations to Other Relations

$=$ is assumed to be a pseudo-relation when related to some non-trivial connective $\bar{\cup}$, $\bar{\subset}$ or $\bar{\cap}$, or to $\equiv$ or $\not\equiv$; consequently, $=$ should not be related to any of these. However, $=$ might be related to itself, and to any "trivial" connective (i.e. in some tautology). $=$ and $\neq$ might also negate one another. When equality cannot be related to some relation, it cannot be equal to it, so that it must be non-equal to it; when equality cannot be related to some relation, it cannot be non-equal to it, so that it must be equal to it; is this consistent or a contradiction?

$\equiv$ (by its two definitions) is related to $\cup$, $\subset$ and $\cap$, and it negates $\not\equiv$.

### 9.0. Symmetry, Transitivity and Reflexivity

$=$ is symmetric, transitive and reflexive; $\neq$ is only symmetric.

$\equiv$ is symmetric, transitive and reflexive; $\not\equiv$ is symmetric and transitive.

### 10.0. Commutation and Permutation

$=$ is commutative and permutative; it is not clear whether $\neq$ is commutative and permutative.

$\equiv$ (by its two definitions) and $\not\equiv$ are commutative and permutative, and they are commutative and permutative "in a way". By the latter I mean that e.g. $\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C}$ equals $\mathcal{A} \equiv \neg\mathcal{B} \equiv \neg\mathcal{C}$, as well as $\neg\mathcal{A} \equiv \neg\mathcal{B} \equiv \mathcal{C}$, as well as $\neg\mathcal{A} \equiv \mathcal{B} \equiv \neg\mathcal{C}$, as well as $\mathcal{A} \not\equiv \mathcal{B} \not\equiv \mathcal{C}$, so that each of these statements is commutative and permutative because $\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C}$ is so.

### 11.0. Tautology

$=$ seems to obey the law of tautology, so that $\mathcal{A} = \mathcal{A}$ and $\mathcal{A} \cup \mathcal{A}$ and $\mathcal{A} \cap \mathcal{A}$ are equal; consequently $\neg(\mathcal{A} = \mathcal{A})$ should equal $\neg\mathcal{A}$, and some difference between $\mathcal{A} = \mathcal{A}$ and $\mathcal{A} = \mathcal{B}$ should exist (and between respectively $\mathcal{A} \cup \mathcal{A}$ and $\mathcal{A} \cup \mathcal{B}$, and $\mathcal{A} \cap \mathcal{A}$ and $\mathcal{A} \cap \mathcal{B}$, when $\mathcal{A} = \mathcal{B}$). The difference is the difference between not corresponding to some connection between "denoting symbols" nor to some connection between "things denoted", and corresponding to some connection between denoting symbols, yet not to some connection between things denoted.

$\equiv$ (by its two definitions) does not obey the law of tautology, for $\mathcal{A} \equiv \mathcal{A}$ and $\mathcal{A} \subset \mathcal{A}$ and $\neg\mathcal{A} \cup \mathcal{A}$ are equal (consequently, any reflexive equivalence is equal to the universal term $\mathbf{U}$). Moreover, $\mathcal{A} \not\equiv \mathcal{A}$ and $\mathcal{A} \cap \neg\mathcal{A}$ are equal (both are contradictions, equal to the empty term $\emptyset$). Note that when reflexive, $\equiv$ and $\not\equiv$ "reduce" to basic relations.

### 12.0. Distribution

$=$ does not involve the law of distribution.

$\equiv$ (by its two definitions) involves the law of distribution. Moreover, the second definition of equivalence (which is of the $\cup$-kind) can be obtained from the first definition (which is of the $\cap$-kind) with the law of distribution.

### 13.0. Association

$=$ obeys the law of association; it is not clear whether $\neq$ obeys the law of association.

$\equiv$ (by its two definitions) obeys the law of association; $\not\equiv$ also obeys the law of association. When either definition of equivalence is used to rewrite a statement like $\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C}$ (with more than one $\equiv$), brackets are required. For example, brackets are required to rewrite any equivalence as some non-equivalence, or vice versa, using either of the two kinds of definition.

### 14.0. Propositional Meaning

$\mathcal{A} = \mathcal{A}$ (as the tautology $\neg\mathcal{A} \subset \mathcal{A}$) says "all $\neg\mathcal{A}$s are $\mathcal{A}$s" and "no $\neg\mathcal{A}$s are $\neg\mathcal{A}$s" and (as the tautology $\mathcal{A} \cap \mathcal{A}$) it says "some $\mathcal{A}$s are $\mathcal{A}$s".

$\mathcal{A} \equiv \mathcal{A}$ (as the reflexive $\mathcal{A} \subset \mathcal{A}$) says "all $\mathcal{A}$s are $\mathcal{A}$s" and "no $\mathcal{A}$s are $\neg\mathcal{A}$s" and (as the reflexive $\neg\mathcal{A} \supset \neg\mathcal{A}$) it says "all $\neg\mathcal{A}$s are $\neg\mathcal{A}$s" and "no $\neg\mathcal{A}$s are $\mathcal{A}$s". Moreover, by the first definition of equivalence, $\mathcal{A} \equiv \mathcal{B}$ says "some "all $\mathcal{A}$s are $\mathcal{B}$s" are "all $\mathcal{B}$s are $\mathcal{A}$s"" and "some "all $\neg\mathcal{B}$s are $\neg\mathcal{A}$s" are "all $\neg\mathcal{A}$s are $\neg\mathcal{B}$s"". Moreover, by the second definition of equivalence, $\mathcal{A} \equiv \mathcal{B}$ says "all "all $\neg\mathcal{A}$s are $\mathcal{B}$s" are "some $\mathcal{A}$s are $\mathcal{B}$s"" and "all "all $\neg\mathcal{B}$s are $\mathcal{A}$s" are "some $\mathcal{B}$s are $\mathcal{A}$s"" and "all "all $\mathcal{A}$s are $\neg\mathcal{B}$s" are "some $\neg\mathcal{A}$s are $\neg\mathcal{B}$s"" and "all "all $\mathcal{B}$s are $\neg\mathcal{A}$s" are "some $\neg\mathcal{B}$s are $\neg\mathcal{A}$s"".

### 15.0. Genesis

$=$, as $=_{df}$, can be used to obtain statements from the negation of the empty term $\emptyset$, given the laws of logic (which too use $=_{df}$), and some number of terms $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, .... Moreover, as $=_v$, $=$ can be used to select some particular statement over others, as soon as these statements are obtained with "genesis" (i.e. from the negation of the empty term, given the laws of logic, and some number of terms).

Any statement with $\equiv$ can be obtained with genesis. Indeed, the universal term $\mathbf{U} = \neg\emptyset$ equals any disjunction of some equivalence and its negation (the non-equivalence). In $\mathbf{U}$, every equivalence equals some disjunction of conjunctions: if the number of terms in the equivalence is $n$, then the number of different disjoint conjunctions should be $2^n \div 2$.

## Appendix A.

In any logical statement of the form $\mathcal{A}\ \mathbf{R}\ \mathcal{B}$, let the bold relation $\mathbf{R}$ be a variable. Potentially, $\mathbf{R}$ is equal to each of the relations $\cup$, $\subset$, $\supset$, $\cap$, $\equiv$, $\not\equiv$, $\neq$ and $=$; in actuality, it equals no relation in particular. As a variable ("per se"), the relation $\mathbf{R}$ is ambiguous.

Call any $\mathcal{A}\ \mathbf{R}\ \mathcal{B}$ with the variable $\mathbf{R}$ a "variable statement", and call the $\mathcal{S}$ that equals the variable statement a "variable non-basic term".

In $\mathcal{A}\ \mathbf{R}\ \mathcal{B}$, let each of the terms $\mathcal{A}$ and $\mathcal{B}$ denote a thing-in-itself (or a situation of things-in-themselves, if it is a non-basic term), and let the relation $\mathbf{R}$ (not bold) denote some connection (say, some degree of likeness versus unlikeness) between the things denoted by $\mathcal{A}$ and $\mathcal{B}$. Let the statement as a whole, and therefore $\mathcal{S}$, denote some situation which consists of the things denoted by $\mathcal{A}$ and $\mathcal{B}$ and the connection denoted by $\mathbf{R}$.

(Alternatively, let $\mathbf{R}$ and $\mathcal{S}$ denote the same connection/situation, so that $\mathbf{R} = \mathcal{S} = (\mathcal{A}\ \mathbf{R}\ \mathcal{B})$, somehow regardless of what $\mathcal{A}$ and $\mathcal{B}$ might denote).

Let $\mathbf{x}$ and $\mathbf{y}$ equal the things denoted by respectively $\mathcal{A}$ and $\mathcal{B}$, and let the semi-colon ; equal the connection denoted by $\mathbf{R}$, and let $[\mathbf{x};\mathbf{y}]$ equal the situation denoted by $\mathcal{S}$ (that is, let these symbols denote the same things as $\mathcal{A}$, $\mathcal{B}$, $\mathbf{R}$ and $\mathcal{S}$; however, treat these symbols as if they are the things denoted by $\mathcal{A}$, $\mathcal{B}$, $\mathbf{R}$ and $\mathcal{S}$). Finally, let | stand for "denotes". Then in short

$$\begin{array}{ccc} \mathcal{A} & \mathbf{R} & \mathcal{B} \\ | & | & | \\ \mathbf{x} & ; & \mathbf{y} \end{array} \quad = \quad \begin{array}{c} (\mathcal{A}\ \mathbf{R}\ \mathcal{B}) \\ | \\ [\mathbf{x};\mathbf{y}] \end{array} \quad = \quad \begin{array}{c} \mathcal{S} \\ | \\ \mathbf{z} \end{array} \quad \text{with } \mathbf{z} = [\mathbf{x};\mathbf{y}]$$

In any logical statement of the form $\mathcal{A}\ \mathrm{R}\ \mathcal{B}$, let the relation R be a particular value of the variable $\mathbf{R}$ (note that in the text, I only use the bold $\mathbf{R}$, even where I should use R). Given $\mathcal{A}\ \mathrm{R}\ \mathcal{B}$, the R must equal some particular relation $\cup$ or $\subset$ or $\supset$ or $\cap$ or $\equiv$ or $\not\equiv$ or $\neq$ or $=$.

When the variable relation $\mathbf{R}$ is somehow "measured"/"observed" or "chosen" to be some particular relation R, the variable statement $\mathcal{A}\ \mathbf{R}\ \mathcal{B}$ should become a "particular statement" $\mathcal{A}\ \mathrm{R}\ \mathcal{B}$, while the variable non-basic term $\mathcal{S}$ should become a "particular non-basic term" $\mathcal{S}$ (not bold).

Now, let "to correspond" mean, either "to denote fully" or "to denote partly" or "to not denote". For example, if in some statement $\mathcal{A}\ \mathrm{R}\ \mathcal{B}$, the term $\mathcal{A}$ corresponds to $\mathbf{x}$, then the $\mathcal{A}$ of this particular statement denotes $\mathbf{x}$ either fully or partly or not at all. Let "to not denote" also mean "to not correspond" and vice versa.

In $\mathcal{A}\ \mathrm{R}\ \mathcal{B}$, let the terms $\mathcal{A}$ and $\mathcal{B}$ correspond to the things denoted by these terms in $\mathcal{A}\ \mathbf{R}\ \mathcal{B}$, and let the relation R (not bold) correspond to the connection denoted by $\mathbf{R}$. Let the statement as a whole, and therefore $\mathcal{S}$ (not bold), correspond to the situation denoted by $\mathcal{A}\ \mathbf{R}\ \mathcal{B}$ as a whole.

Let $\vdots$ stand for "corresponds to". Then in short

$$\begin{array}{ccccccccc} \mathcal{A} & \mathrm{R} & \mathcal{B} & & (\mathcal{A}\ \mathrm{R}\ \mathcal{B}) & & \mathcal{S} & & \\ \vdots & \vdots & \vdots & = & \vdots & = & \vdots & \text{with} & \mathbf{z} = [\mathbf{x};\mathbf{y}] \\ \mathbf{x} & ; & \mathbf{y} & & [\mathbf{x};\mathbf{y}] & & \mathbf{z} & & \end{array}$$

Let any particular relation R equal some "act of selection", and let any particular statement $(\mathcal{A}\ \mathrm{R}\ \mathcal{B}) = \mathcal{S}$ denote (select) some "selection" from the situation to which it corresponds. In short

$$\begin{array}{cccccc} \mathcal{A}\ \mathrm{R}\ \mathcal{B} & & (\mathcal{A}\ \mathrm{R}\ \mathcal{B}) & & \mathcal{S} & \\ & = & | & = & | & \text{with}\ \ \mathbf{s} = \text{some selection.} \\ & & \mathbf{s} & & \mathbf{s} & \end{array}$$

(Alternatively, let R and $\mathcal{S}$ denote the same act of selection/selection, so that $\mathrm{R} = \mathcal{S} = (\mathcal{A}\ \mathrm{R}\ \mathcal{B})$, somehow regardless of what $\mathcal{A}$ and $\mathcal{B}$ might denote).

For example, $\mathcal{A} \cup \mathcal{B}$, equal to $(\mathcal{A} \cap \neg\mathcal{B}) \cup (\mathcal{A} \cap \mathcal{B}) \cup (\neg\mathcal{A} \cup \mathcal{B})$, denotes (selects) what both $\mathcal{A}$ and $\neg\mathcal{B}$ have in common, and/or what both $\mathcal{A}$ and $\mathcal{B}$ have in common, and/or what both $\neg\mathcal{A}$ and $\mathcal{B}$ have in common.

$$\begin{array}{ccccccccccc} \mathcal{A} & \cup & \mathcal{B} & & (\mathcal{A} \cup \mathcal{B}) & & \mathcal{S} & & (\mathcal{A} \cup \mathcal{B}) & & \mathcal{S} \\ \vdots & \vdots & \vdots & = & \vdots & = & \vdots & \text{equals} & | & = & | \\ \mathbf{x} & ; & \mathbf{y} & & [\mathbf{x};\mathbf{y}] & & \mathbf{z} & & \mathbf{s} & & \mathbf{s} \end{array}$$

For example, $\mathcal{A} \cap \mathcal{B}$ denotes (selects) what both $\mathcal{A}$ and $\mathcal{B}$ have in common.

$$\begin{array}{ccccccccccc} \mathcal{A} & \cap & \mathcal{B} & & (\mathcal{A} \cap \mathcal{B}) & & \mathcal{S} & & (\mathcal{A} \cap \mathcal{B}) & & \mathcal{S} \\ \vdots & \vdots & \vdots & = & \vdots & = & \vdots & \text{equals} & | & = & | \\ \mathbf{x} & ; & \mathbf{y} & & [\mathbf{x};\mathbf{y}] & & \mathbf{z} & & \mathbf{s} & & \mathbf{s} \end{array}$$

For example, $\mathcal{A} \equiv \mathcal{B}$, equal to $(\mathcal{A} \cap \mathcal{B}) \cup (\neg\mathcal{A} \cap \neg\mathcal{B})$, denotes (selects) what both $\mathcal{A}$ and $\mathcal{B}$ have in common, and/or what both $\neg\mathcal{A}$ and $\neg\mathcal{B}$ have in common.

$$\begin{array}{ccccccccccc} \mathcal{A} & \equiv & \mathcal{B} & & (\mathcal{A} \equiv \mathcal{B}) & & \mathcal{S} & & (\mathcal{A} \equiv \mathcal{B}) & & \mathcal{S} \\ \vdots & \vdots & \vdots & = & \vdots & = & \vdots & \text{equals} & | & = & | \\ \mathbf{x} & ; & \mathbf{y} & & [\mathbf{x};\mathbf{y}] & & \mathbf{z} & & \mathbf{s} & & \mathbf{s} \end{array}$$

Next consider $\mathcal{A} \neq \mathcal{B}$. Does this statement only correspond to what $\mathcal{A}\ \mathbf{R}\ \mathcal{B}$ denotes, or does it denote the same as $\mathcal{A}\ \mathbf{R}\ \mathcal{B}$ (so that in $\mathcal{A} \neq \mathcal{B}$, the term $\mathcal{A}$ always fully denotes $\mathbf{x}$, and $\mathcal{B}$ always fully denotes $\mathbf{y}$, and $\neq$ denotes the same as the variable $\mathbf{R}$, and $\mathbf{z} = \mathbf{s}$)? That is, ask

$$\begin{array}{ccccccccccc} \mathcal{A} & \neq & \mathcal{B} & & (\mathcal{A} \neq \mathcal{B}) & & \mathcal{S} & & (\mathcal{A} \neq \mathcal{B}) & & \mathcal{S} \\ | & | & | & = & | & = & | & \text{equals?} & | & = & | \\ \mathbf{x} & ; & \mathbf{y} & & [\mathbf{x};\mathbf{y}] & & \mathbf{z} & & \mathbf{s} & & \mathbf{s} \end{array}$$

If the relation $\neq$ and the variable $\mathbf{R}$ cannot be equal (which, after all, might turn either the value $=$ or the variable $\mathbf{R}$ into a contradiction), the question is

$$\begin{array}{ccc} \mathcal{A} & \neq & \mathcal{B} \\ | & \vdots & | \\ \mathbf{x} & ; & \mathbf{y} \end{array} = \begin{array}{c} (\mathcal{A} \neq \mathcal{B}) \\ \vdots \\ [\mathbf{x};\mathbf{y}] \end{array} = \begin{array}{c} \mathcal{S} \\ \vdots \\ \mathbf{z} \end{array} \text{ equals? } \begin{array}{c} (\mathcal{A} \neq \mathcal{B}) \\ | \\ \mathbf{s} \end{array} = \begin{array}{c} \mathcal{S} \\ | \\ \mathbf{s} \end{array}$$

Note that if this relation $\neq$ is identical to the relation $\bar{\cup}$ or to the relation $\not\equiv$ (see appendix C), then $\bar{\cup}$ or $\not\equiv$ may replace $\neq$ in this question.

Next consider $\mathcal{A} = \mathcal{B}$. The theory is that the terms $\mathcal{A}$ and $\mathcal{B}$ denote the same thing (say, $\mathbf{x}$); therefore $=$ does not correspond to some connection between things denoted. Accordingly

$$\begin{array}{ccc} \mathcal{A} & = & \mathcal{B} \\ | & & | \\ \mathbf{x} & & \mathbf{x} \end{array} = \begin{array}{c} (\mathcal{A} = \mathcal{B}) \\ | \\ \mathbf{x} \end{array} = \begin{array}{c} \mathcal{S} \\ | \\ \mathbf{x} \end{array} \text{ equals } \begin{array}{c} (\mathcal{A} = \mathcal{B}) \\ | \\ \mathbf{s} \end{array} = \begin{array}{c} \mathcal{S} \\ | \\ \mathbf{s} \end{array}$$

Replace $\mathcal{A}$ by $\neg\mathcal{A}$ and $\mathcal{B}$ by $\neg\mathcal{B}$ to obtain the negation of this pseudo-statement.

The theory of denotation of this appendix is rather crude; I intend to improve upon it somewhere else (for example, I intend to clarify the distinction between "things denoting" and "things denoted", and why a special kind of relation called "denotation" should exist as distinct from both equality and equivalence). For now, the appendix and the first section must do.

## Appendix B.

Given some equivalence $\mathcal{A} \equiv \mathcal{B}$, there are two kinds of definition. The "first definition" $(\mathcal{A} \subset \mathcal{B}) \cap (\mathcal{A} \supset \mathcal{B})$ is of the $\cap$-kind (because the middle relation is conjunction). The "second definition" $(\mathcal{A} \cup \mathcal{B}) \subset (\mathcal{A} \cap \mathcal{B})$ is of the $\cup$-kind (because the middle relation can be changed into disjunction with the identities named after De Morgan). Of each kind of definition, there are sixty-four equal forms (by the identities named after De Morgan). Moreover, the second definition (the $\cup$-kind) can be obtained from the first (the $\cap$-kind) with the law of distribution.

By the first definition, $\mathcal{A} \equiv \mathcal{B}$ is equal to the $4 \bullet 4 = 16$ statements

$$\neg \left| \begin{array}{lcr} \neg(\neg\mathcal{A} \cup \mathcal{B}) & & \neg(\mathcal{A} \cup \neg\mathcal{B}) \\ = & & = \\ \neg(\mathcal{A} \subset \mathcal{B}) & & \neg(\neg\mathcal{A} \subset \neg\mathcal{B}) \\ = & \cup & = \\ \neg(\neg\mathcal{A} \supset \neg\mathcal{B}) & & \neg(\mathcal{A} \supset \mathcal{B}) \\ = & & = \\ \mathcal{A} \cap \neg\mathcal{B} & & \neg\mathcal{A} \cap \mathcal{B} \end{array} \right|$$

equal to the $4 \bullet 4 = 16$ statements

$$\neg \left| \begin{array}{lcr} \neg\mathcal{A} \cup \mathcal{B} & & \neg(\mathcal{A} \cup \neg\mathcal{B}) \\ = & & = \\ \mathcal{A} \subset \mathcal{B} & & \neg(\neg\mathcal{A} \subset \neg\mathcal{B}) \\ = & \subset & = \\ \neg\mathcal{A} \supset \neg\mathcal{B} & & \neg(\mathcal{A} \supset \mathcal{B}) \\ = & & = \\ \neg(\mathcal{A} \cap \neg\mathcal{B}) & & \neg\mathcal{A} \cap \mathcal{B} \end{array} \right|$$

equal to the $4 \bullet 4 = 16$ statements

$$\neg \left| \begin{array}{lcr} \neg(\neg\mathcal{A} \cup \mathcal{B}) & & \mathcal{A} \cup \neg\mathcal{B} \\ = & & = \\ \neg(\mathcal{A} \subset \mathcal{B}) & & \neg\mathcal{A} \subset \neg\mathcal{B} \\ = & \supset & = \\ \neg(\neg\mathcal{A} \supset \neg\mathcal{B}) & & \mathcal{A} \supset \mathcal{B} \\ = & & = \\ \mathcal{A} \cap \neg\mathcal{B} & & \neg(\neg\mathcal{A} \cap \mathcal{B}) \end{array} \right|$$

equal to the $4 \bullet 4 = 16$ statements

$$\begin{array}{lcr} \neg\mathcal{A} \cup \mathcal{B} & & \mathcal{A} \cup \neg\mathcal{B} \\ = & & = \\ \mathcal{A} \subset \mathcal{B} & & \neg\mathcal{A} \subset \neg\mathcal{B} \\ = & \cap & = \\ \neg\mathcal{A} \supset \neg\mathcal{B} & & \mathcal{A} \supset \mathcal{B} \\ = & & = \\ \neg(\mathcal{A} \cap \neg\mathcal{B}) & & \neg(\neg\mathcal{A} \cap \mathcal{B}) \end{array}$$

Negate these four sets of equal statements to obtain all sixty-four equal forms of the second definition of $\mathcal{A} \not\equiv \mathcal{B}$ (the negation of $\mathcal{A} \equiv \mathcal{B}$).

By the second definition, $\mathcal{A} \equiv \mathcal{B}$ is equal to the 4•4=16 statements

$$\begin{array}{lcr} \neg(\mathcal{A} \cup \mathcal{B}) & & \neg(\neg\mathcal{A} \cup \neg\mathcal{B}) \\ = & & = \\ \neg(\neg\mathcal{A} \subset \mathcal{B}) & & \neg(\mathcal{A} \subset \neg\mathcal{B}) \\ = & \cup & = \\ \neg(\mathcal{A} \supset \neg\mathcal{B}) & & \neg(\neg\mathcal{A} \supset \mathcal{B}) \\ = & & = \\ \neg\mathcal{A} \cap \neg\mathcal{B} & & \mathcal{A} \cap \mathcal{B} \end{array}$$

equal to the $4 \bullet 4 = 16$ statements

$$\begin{array}{lcr} \mathcal{A} \cup \mathcal{B} & & \neg(\neg\mathcal{A} \cup \neg\mathcal{B}) \\ = & & = \\ \neg\mathcal{A} \subset \mathcal{B} & & \neg(\mathcal{A} \subset \neg\mathcal{B}) \\ = & \subset & = \\ \mathcal{A} \supset \neg\mathcal{B} & & \neg(\neg\mathcal{A} \supset \mathcal{B}) \\ = & & = \\ \neg(\neg\mathcal{A} \cap \neg\mathcal{B}) & & \mathcal{A} \cap \mathcal{B} \end{array}$$

equal to the $4 \bullet 4 = 16$ statements

$$\begin{array}{lcr} \neg(\mathcal{A} \cup \mathcal{B}) & & \neg\mathcal{A} \cup \neg\mathcal{B} \\ = & & = \\ \neg(\neg\mathcal{A} \subset \mathcal{B}) & & \mathcal{A} \subset \neg\mathcal{B} \\ = & \supset & = \\ \neg(\mathcal{A} \supset \neg\mathcal{B}) & & \neg\mathcal{A} \supset \mathcal{B} \\ = & & = \\ \neg\mathcal{A} \cap \neg\mathcal{B} & & \neg(\mathcal{A} \cap \mathcal{B}) \end{array}$$

equal to the $4 \bullet 4 = 16$ statements

$$\neg \left| \begin{array}{lcr} \mathcal{A} \cup \mathcal{B} & & \neg\mathcal{A} \cup \neg\mathcal{B} \\ = & & = \\ \neg\mathcal{A} \subset \mathcal{B} & & \mathcal{A} \subset \neg\mathcal{B} \\ = & \cap & = \\ \mathcal{A} \supset \neg\mathcal{B} & & \neg\mathcal{A} \supset \mathcal{B} \\ = & & = \\ \neg(\neg\mathcal{A} \cap \neg\mathcal{B}) & & \neg(\mathcal{A} \cap \mathcal{B}) \end{array} \right|$$

Negate these four sets of equal statements to obtain all sixty-four equal forms of the first definition of $\mathcal{A} \not\equiv \mathcal{B}$ (the negation of $\mathcal{A} \equiv \mathcal{B}$).

Indeed, like any disjunction equals the negation of some conjunction, and vice versa, the definitions of the $\cap$-kind and $\cup$-kind of $\mathcal{A} \equiv \mathcal{B}$, negate the definitions of, respectively, the $\cup$-kind and $\cap$-kind of $\mathcal{A} \not\equiv \mathcal{B}$.

Note that in this appendix |...| is used for bracketing instead of (...).

## Appendix C.

In this work, $=$ is "full equality", so that in $\mathcal{A} = \mathcal{B}$, all of $\mathcal{A}$ is equal to all of $\mathcal{B}$, and vice versa. On the other hand, $\neq$ might mean "full non-equality" or "partial non-equality", and if $\mathcal{A}$ and $\mathcal{B}$ in $\mathcal{A} \neq \mathcal{B}$ are partial non-equal, then all of $\mathcal{A}$ might be non-equal to some of $\mathcal{B}$, or some of $\mathcal{A}$ might be non-equal to all of $\mathcal{B}$, or only some of $\mathcal{A}$ might be non-equal to only some of $\mathcal{B}$.

Perhaps, when it corresponds to some connection between things-in-themselves, let the symbol $\neq$ equal the symbol $\bar{\cup}$, which stands for non-trivial disjunction (i.e. any disjunction except a tautological one). Then non-equality corresponds to the same connection between things-in-themselves as proper disjunction does.

The terms of any non-trivial disjunction $\mathcal{A} \mathbin{\bar{\cup}} \mathcal{B}$ are either fully or partly non-equal, and when partly non-equal, the one term is either fully included in the other, or the terms are partly included in one another. Consequently the same goes for the terms of any non-equality $\mathcal{A} \neq \mathcal{B}$ that equals $\mathcal{A} \mathbin{\bar{\cup}} \mathcal{B}$.

Of course, when $\mathcal{A} \neq \mathcal{B}$ is equal to a non-trivial $\mathcal{A} \mathbin{\bar{\cup}} \mathcal{B}$, its negation $\neg(\mathcal{A} \neq \mathcal{B})$ is equal to $\neg(\mathcal{A} \mathbin{\bar{\cup}} \mathcal{B})$, equal to $\neg\mathcal{A} \cap \neg\mathcal{B}$; this conjunction is not equal to $\mathcal{A} = \mathcal{B}$ unless both statements are contradictions, equal to the empty-term $\emptyset$.

Note that if we replace $\cup$ by $\neq$ in the trivial tautology $(\mathcal{A} \cup \mathcal{A}) = \mathcal{A}$, we obtain $(\mathcal{A} \neq \mathcal{A}) = \mathcal{A}$, which is a contradiction; for this reason use $\neq \, = \, \bar{\cup}$.

Also note that neither $\mathcal{A} = \neg\mathcal{B}$ nor $\neg\mathcal{A} = \mathcal{B}$ equals $\mathcal{A} \neq \mathcal{B}$, and compare this to $(\mathcal{A} \equiv \neg\mathcal{B}) = (\neg\mathcal{A} \equiv \mathcal{B}) = (\mathcal{A} \not\equiv \mathcal{B})$ (see below).

Finally, one might wonder why/when to choose $\neq \, = \, \bar{\cup}$ over $\neq \, = \, \not\equiv$, and why not $\not\equiv \, = \, \bar{\cup}$?

The reason is that in $\mathcal{A} \mathbin{\bar{\cup}} \mathcal{B}$, the terms $\mathcal{A}$ and $\mathcal{B}$ are either fully or partly non-equal, and when partly non-equal, the one term is either fully or partly included in the other; however, in $\mathcal{A} \not\equiv \mathcal{B}$, the terms $\mathcal{A}$ and $\mathcal{B}$ are either fully or partly non-equal, and in either case the one term is fully included in the other (in the first case, somehow as the empty-term $\emptyset$).

To see why, recall that $\mathcal{A} \not\equiv \mathcal{B}$ negates $\mathcal{A} \equiv \mathcal{B}$, which by the second definition of equivalence (and the identities named after De Morgan, the law of distribution, and the fact that any contradiction like $\mathcal{A} \cap \neg\mathcal{A}$ denotes nothing) equals $\neg((\mathcal{A} \cap \neg\mathcal{B}) \cup (\mathcal{B} \cap \neg\mathcal{A}))$. Hence (by the law of double negation) $\mathcal{A} \not\equiv \mathcal{B}$ equals $(\mathcal{A} \cap \neg\mathcal{B}) \cup (\mathcal{B} \cap \neg\mathcal{A})$.

Now in $(\mathcal{A} \cap \neg\mathcal{B}) \cup (\mathcal{B} \cap \neg\mathcal{A})$ there are two conjunctions. The first conjunction is equal to $\neg(\mathcal{A} \subset \mathcal{B}) \cap (\mathcal{A} \supset \mathcal{B})$ which says "$\mathcal{B}$ fully implies/is included in $\mathcal{A}$, and not vice versa"; the second conjunction is equal to $(\mathcal{A} \subset \mathcal{B}) \cap \neg(\mathcal{A} \supset \mathcal{B})$ which says "$\mathcal{A}$ fully implies/is included in $\mathcal{B}$, and not vice versa". Indeed

$\neg(\mathcal{A} \subset \mathcal{B}) \cap (\mathcal{A} \supset \mathcal{B})$ is equal to
$\neg(\neg\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup \neg\mathcal{B})$, equal to
$(\mathcal{A} \cap \neg\mathcal{B}) \cap (\mathcal{A} \cup \neg\mathcal{B})$, equal to
$(\mathcal{A} \cap \neg\mathcal{B} \cap \mathcal{A}) \cup (\mathcal{A} \cap \neg\mathcal{B} \cap \neg\mathcal{B})$, equal to
$(\mathcal{A} \cap \neg\mathcal{B}) \cup (\mathcal{A} \cap \neg\mathcal{B})$, i.e. to $\mathcal{A} \cap \neg\mathcal{B}$

$(\mathcal{A} \subset \mathcal{B}) \cap \neg(\mathcal{A} \supset \mathcal{B})$ is equal to
$(\neg\mathcal{A} \cup \mathcal{B}) \cap \neg(\mathcal{A} \cup \neg\mathcal{B})$, equal to
$(\neg\mathcal{A} \cup \mathcal{B}) \cap (\neg\mathcal{A} \cap \mathcal{B})$, equal to
$(\neg\mathcal{A} \cap \neg\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{B} \cap \neg\mathcal{A} \cap \mathcal{B})$, equal to
$(\neg\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{B} \cap \neg\mathcal{A})$, i.e. to $\neg\mathcal{A} \cap \mathcal{B}$

In addition, note that $\mathcal{A} \cup \mathcal{B}$ is considered equal to the computational table

| | $\mathcal{B}$ | $\neg\mathcal{B}$ |
|---|---|---|
| $\mathcal{A}$ | *yes* | *yes* |
| $\neg\mathcal{A}$ | *yes* | *no* |

$$= (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \neg\mathcal{B}) \cup (\neg\mathcal{A} \cap \mathcal{B}) \cup \cancel{(\neg\mathcal{A} \cap \neg\mathcal{B})}$$

This statement is equal to $(\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \neg\mathcal{B}) \cup (\neg\mathcal{A} \cap \mathcal{B})$, equal to $(\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \not\equiv \mathcal{B})$.

Accordingly, the computational gate of $\mathcal{A} \cup \mathcal{B}$ could provide output when its input is provided by either the computational gate of $\mathcal{A} \cap \mathcal{B}$ or that of $\mathcal{A} \not\equiv \mathcal{B}$ or both.

## Appendix D.

Consider $\mathcal{A} \cap_{\mathfrak{p}} \neg\mathcal{B}$ equal to $\mathcal{A} -_{\mathfrak{p}} \mathcal{B}$ (as well as to $\neg\mathcal{B} -_{\mathfrak{p}} \neg\mathcal{A}$).

Consider $\neg\mathcal{A} \cap_{\mathfrak{p}} \mathcal{B}$ equal to $\mathcal{B} -_{\mathfrak{p}} \mathcal{A}$ (as well as to $\neg\mathcal{A} -_{\mathfrak{p}} \neg\mathcal{B}$).

Let $\mathfrak{a}$ and $\mathfrak{b}$ be two cardinal numbers.

Let $\mathfrak{a} - \mathfrak{b}$ equal $\mathcal{A} -_{\mathfrak{c}} \mathcal{B}$.

Let $\mathfrak{b} - \mathfrak{a}$ equal $\mathcal{B} -_{\mathfrak{c}} \mathcal{A}$.

Let $\mathfrak{a} > \mathfrak{b}$ equal $\neg(\mathcal{A} \subset_{\mathfrak{c}} \mathcal{B}) \cap (\mathcal{A} \supset_{\mathfrak{c}} \mathcal{B})$.

Let $\mathfrak{a} < \mathfrak{b}$ equal $(\mathcal{A} \subset_{\mathfrak{c}} \mathcal{B}) \cap \neg(\mathcal{A} \supset_{\mathfrak{c}} \mathcal{B})$.

Then $\mathfrak{a} > \mathfrak{b}$ is equal to $\mathfrak{a} - \mathfrak{b}$ (i.e. some of $\mathfrak{a}$ is not $\mathfrak{b}$).

Then $\mathfrak{a} < \mathfrak{b}$ is equal to $\mathfrak{b} - \mathfrak{a}$ (i.e. some of $\mathfrak{b}$ is not $\mathfrak{a}$).

Then $(\mathfrak{a} < \mathfrak{b}) \cap (\mathfrak{a} > \mathfrak{b})$ is a contradiction (the ancient reduction ad absurdum).

Then $\neg(\mathfrak{a} < \mathfrak{b}) \cap \neg(\mathfrak{a} > \mathfrak{b})$ equals $\mathcal{A} \equiv_{\mathfrak{c}} \mathcal{B}$.

## Appendix E.

Naturally, after I finalized the first (somewhat imperfect) draft of this paper (in early 2021), I tried to publish it. Since the mathematics and philosophy are pretty orthodox and basic, I assumed the paper would be understood, and met with at least a little interest; perhaps it would be challenged, and I would be able to reply to these challenges, and so on. Instead, I encountered disinterest, dislike and/or disdain; probably I just did not find the right outlet. Anyhow, one mathematical journal was kind enough to send me a relatively useful report; here, I take the liberty to address it. The aim is to make the work less inaccessible and to better explain its relevance, for in these regards I seem to have failed.

The report mentions the following perceived problems (regretfully I am not allowed to give the report in verbatim)

- It is clear that "equivalence" means "logical equivalence" in the sense of "bi-implication"; it is not at all clear what "equality" means.
- No formal logical system is ever specified.
- Most citations are to non-contemporary works.
- In contemporary logic, like "first-order logic", "higher-order logic", "simply type theory", "dependent type theory", etc. the kinds of objects to which the logic applies are clearly specified, so that the confusing vagueness of "thing-in-itself" can be left out.
- In contemporary logic, equality and equivalence are also clearly specified. For example, in "first-order logic" only elements are compared with equality and only other formulas with the connective equivalence, so the use of these relations depends on what is compared. Moreover, in some forms of "higher-order logic" and "type theory" equality coincides with equivalence, like in "multi-sorted higher-order logic with a sort of propositions", and like in "dependent type theory" when "propositional univalence" is used.
- When equality and equivalence of formulas do not coincide, nothing meaningful can be said. So there is no reason to compare the two. There are no new mathematical results in the paper, nor philosophical results. Therefore publication is not recommended. [Problem solved; sleep safe and sound.]

OK, call the "formal logical system that is never specified" system $\mathcal{A}\ \mathbf{R}\ \mathcal{B}$, or say $\mathcal{A}\ \mathbf{R}\ \mathcal{B}_{system}$; it is the most general system there is (as is mentioned in 1.1.; of course, this system goes beyond binary statements). Depending on what relations and terms (including "elements", including statements, hence "formulas", and so on) are chosen for the placeholders $\mathcal{A}$, $\mathcal{B}$ and $\mathbf{R}$, we could have for example $\mathcal{A}\ \mathbf{R}\ \mathcal{B}_{system=1st\ order\ logic}$, or for example $\mathcal{A}\ \mathbf{R}\ \mathcal{B}_{system=nth\ order\ logic}$, or for example $\mathcal{A}\ \mathbf{R}\ \mathcal{B}_{system=simple\ type\ theory}$, or for example $\mathcal{A}\ \mathbf{R}\ \mathcal{B}_{system=n\ type\ theory}$; we could even design a system in which statements from different systems are compared, for example $\mathcal{A}\ \mathbf{R}\ \mathcal{B}_{system=math\ \mathbf{R}\ metamath}$, with $\mathcal{A} = (\mathcal{A}\ \mathbf{R}\ \mathcal{B})_{system=math}$ and $\mathcal{B} = (\mathcal{A}\ \mathbf{R}\ \mathcal{B})_{system=metamath}$. In this sense, there is only one system for us to be concerned with in mathematics as a whole, namely $\mathcal{A}\ \mathbf{R}\ \mathcal{B}$. It is that general (and being so, it is indeed not specific).

Now in this work, as recognized, I compare equality with logical equivalence; the latter being defined in terms of the logical connectives (I left out the Cartesian product). Please recall that classical computer architecture relies on the logical connectives in the form of gates, so that literally anything you can do on a classical computer (including building whatever kind of logical system in whatever kind of language) ultimately translates into the system used in this work. Again, it is that general.

By equality I mean the relation used for permanent definitions and impermanent definitions (where some variable is put equal to some value). As such, the relation pertains not only to logic, but to mathematics as a whole, and any other field that makes use of definitions. In my eyes, logical equivalence is just as universal (and so seen, is rarely used well in mathematics, physics, or philosophy; people tend to rely on intuition, without giving much attention to the consequences that result from the definition); however, if that is not agreeable, one could compare equality with any other non-trivial relation and maintain the "heart of the matter", which is as follows.

To say e.g. "what the author means by equality is not clear" is justified. However, the perceived problem does not lie with me; it lies with the nature of equality. Equality is enigmatic; intuitively as weird as how liquid helium flows at almost zero degrees Kelvin. No logical system can escape from the enigmatic nature (theorems like "propositional univalence" do not do so, see below); this is what some founders understood, and for some reason seems to have been forgotten. To be more clear, let me "clearly specify" equality, say, by defining it in terms of some "well-behaved" relation $\mathbf{R}_=$, so that $(\mathcal{A} = \mathcal{B}) = (\mathcal{A}\,\mathbf{R}_=\mathcal{B})$.

Now if $(\mathcal{A} = \mathcal{B}) = (\mathcal{A}\,\mathbf{R}_=\mathcal{B})$

Then $((\mathcal{A} = \mathcal{B}) = (\mathcal{A}\,\mathbf{R}_=\mathcal{B})) = ((\mathcal{A} = \mathcal{B})\,\mathbf{R}_=(\mathcal{A}\,\mathbf{R}_=\mathcal{B}))$

And $(((\mathcal{A} = \mathcal{B}) = (\mathcal{A}\,\mathbf{R}_=\mathcal{B})) = ((\mathcal{A} = \mathcal{B})\,\mathbf{R}_=(\mathcal{A}\,\mathbf{R}_=\mathcal{B}))) =$
$(((\mathcal{A} = \mathcal{B}) = (\mathcal{A}\,\mathbf{R}_=\mathcal{B}))\,\mathbf{R}_=\,((\mathcal{A} = \mathcal{B})\,\mathbf{R}_=(\mathcal{A}\,\mathbf{R}_=\mathcal{B})))$

And *et cetera*

This infinite regress can only be avoided if $\mathbf{R}_=$ is trivial/tautological, and equality as used in this paper is maintained. One could also choose not to avoid it; in that case equality as used in this paper does not exist, and we would have a logic without this equality. One problem with such a logic is that you cannot apply the identities named after De Morgan as is done in this paper (instead you would have, say, "De Morgan equivalences"). However, only accepting, say, "De Morgan equivalences" kind of sabotages the whole field of computer architecture, i.e. the fact that some scheme of gates (implementing conjunctions, and disjunctions and thereby, it is believed, inclusions) could be fully replaced by some other scheme which might yield faster results (we do not have to connect the two schemes by some implementation of equivalence); in other words, it denies the reality we live in (the age of computing).

Finally, let me say something about the concept of "thing-in-itself". In my eyes, denying it also amounts to denying reality; however, to properly make this case requires an entire paper in its own right, which I intend to produce at some later point (of course one could just study a little philosophy and/or neurobiology to understand its raison d'être). Since the concept is taken for granted in this work, attacking it is justified. On the other hand, what's in a name? To understand equality as used in this paper, one has to invent some kind of "mapping" called "denotation" (which also really needs to be explained better, for example what is this mapping, or mapping in general, in computer architecture?), and then you could just have two sets of terms – denoting and denoted – and the latter would be "thing-in-itself" with respect to the former. Philosophically, I am still not that pleased; I prefer the approach by Schopenhauer, who argues that there is only one thing-in-itself, which is variable-like or choice-function-like and called "will" (he calls the denoting terms "ideas"); however, right now, I do not have the space to address these matters properly; yes, this is where the paper falls short, severely so (for example, I did not even address the tautological infinite regress in e.g. figure 3).

To conclude, indeed, my paper fails to say much that is new, and yet it does not; such is the tragedy of contemporary logic. For the human mind, the field of logic just happens to be a labyrinth and is still far from fully understood; I think that part of "properly modern logic" lost the thread without knowing it, and without need, considering the quality of the foundations of mathematical logic (Nietzsche would have loved this). I am confident that when the time of understanding comes, a very simple form of logic will be readily applicable to literally anything (Nietzsche would have hated that), not just classical computing. Getting a little closer to that point is my agenda.

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